Diagonal Games: A Tool for Experiments and Theory [*]

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Abstract

This paper introduces diagonal games, a new class of two-player dominance-solvable games which constitutes a useful benchmark in the study of cognitive limitations in strategic settings, both for exploring predictions of theoretical models and for experiments. This class of finite games allows for a disciplined way to vary two features of the strategic setting plausibly related to game complexity: the number of steps of iterated elimination of dominated actions required to reach the dominance solution and the number of actions. Furthermore, I derive testable implications of solution concepts such as level-k, endogenous depth of reasoning, sampling equilibrium, and quantal response equilibrium.

Keywords: Diagonal Games; Dominance; Level-k; Strategic Complexity; Bounded Rationality.

JEL Classifications: C72, C92.

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Thinking is costly. Moreover, individuals make worse choices in decision problems that are more complex, arguably entailing greater cognitive costs. This is has not only been documented in lab settings (Oprea 2020), but is also a standard explanation for real-world phenomena, for instance, with cognitive costs being suggested as an explanation for firms' obfuscation practices (e.g. Grubb 2015). As thinking is costly and cognitive costs and computational complexity play an important role in determining choices, this aspect has been recently incorporated in decision-making models (e.g. Ortoleva 2013; Caplin and Dean 2015; Camara 2020). Similarly, different game-theoretic models have been put forth emphasizing cognitive limitations (Stahl and Wilson 1995; Nagel 1995) and computational costs (Halpern and Pass 2015; Alaoui and Penta 2016) as key elements underlying deviations of gameplay from Nash equilibrium.

However, there is no disciplined way to manipulate the complexity of a strategic setting in an unambiguous manner while keeping the environment fundamentally unchanged. While it is natural to consider the number of alternatives as a proxy for computational complexity — and indeed it works well in many classes of decision problems —, simply expanding a player's action set can change the nature of the strategic interaction. For instance, it can turn a dominance-solvable game into a game with multiple Nash equilibria or vice-versa. Furthermore, it can reasonably affect relevant but not modelled experimental considerations as rendering specific outcomes more or less salient or affect their perceived fairness. Having a class of games allowing for a disciplined way to vary the environment's complexity provides a valuable benchmark, as it enables both experimenters to document new patterns within a comparable class of games and theorists to explore testable implications of new and existing solution concepts.

This paper introduces a new class of finite normal-form games which we call diagonal games, a class of two-player, dominance-solvable games. The games are characterized by a pair of single-peaked functions defined on the integer numbers and by the size of the action sets. In this class of games, the number of steps of iterated elimination of dominated actions needed to reach the dominance solution is pinned down by the number of actions and the maximizer of the functions as are the number of actions deleted. I provide a partial order on diagonal games that I argue captures a natural understanding of increasing complexity within this class. I then derive testable implications and comparative statics for a number of solution concepts with respect to such complexity ranking, illustrating the usefulness of diagonal games as a benchmark for investigating the effects of complexity on gameplay. Furthermore, I present and explore the properties of two variants of this class of games, one with a continuum of actions and another which turns the dominance-solvable game into a game with a unique symmetric Nash equilibrium in mixed strategies.

In diagonal games, the payoff structure provides a strategic ranking of actions, a sensible linear order on the action space relating to iterated dominance. I argue for a specific notion of a game being more complex than another by taking into account the size of the action set and the number of steps to reach the dominance solution. These two elements are used to explore the implications of the level-k model (Stahl and Wilson 1995; Nagel 1995), endogenous depth of reasoning (Alaoui and Penta 2016) which endogenizes level-k reasoning via cost-benefit considerations, Osborne and Rubinstein's (2003) sampling equilibrium where players observe a fixed number of samples from their opponents' equilibrium distribution of actions — a proxy for noisy belief formation. I further consider quantal response equilibrium and its rank-dependent choice generalization (McKelvey and Palfrey 1995; Goeree et al. 2019), which captures payoff-dependent mistakes.

I show that, in this class of games, actions taken by level-k players are identified in a manner which is robust to risk-attitudes and monotone in complexity with respect to the actions' strategic ranking. With respect to the endogenous depth of reasoning model, where player's level of reasoning results from trading off the gain and cost of reasoning, I provide clear comparative statics: actions taken are monotone with respect to the actions' strategic ranking in the cost of reasoning, the payoff scale and complexity. I prove similar monotonicity results of sampling equilibrium gameplay in the game's complexity and establish specific testable predictions implied by rank-dependent choice.

Two variants of this class of games are introduced and analyzed. The framework of diagonal games naturally extends to games where the action set corresponds to a closed interval in the real line as opposed to a finite set. In this case, the measure of the interval takes the place of the number of actions and the results regarding the general properties of the game, the level-k and the endogenous depth of reasoning model also hold. Finally, a small modification on the properties the functions determining the payoffs — corresponding to a single payoff out — leads to all Nash equilibria being in (non-degenerate) mixed-strategies and, in symmetric games, to a unique symmetric Nash equilibrium. A discussion on the predictions by the models mentioned above ensues.

Beyond its usefulness as a benchmark environment for theory, this class of games was designed with experimental implementations in mind. First, it provides a scalable and flexible benchmark of two-player dominance-solvable games. This is of practical interest as there is no readily available class of finite dominance-solvable games to support experimenters in their design choices despite the fact that possibly many experiments involve games of this type.[1] On top of increased design difficulties, this resulted in each paper presenting multiple payoff structures often disagreeing in the order and number of actions deleted at each step of deletion of dominated actions, complicating the analysis or comparisons across different experiments and even across games within the same experiment. Beyond their potential use as a benchmark environment for theory, diagonal games also constitute an elegant solution for this practical problem.

Second, this class of games was designed in a way intended to avoid or mitigate other potential experimental concerns. In diagonal games, the payoff to the dominance solution is neither strictly Pareto dominant nor Pareto dominated. Moreover, the payoffs associated to the dominance solution are not unique but instead appear multiple times in the bimatrix. This fact together with randomization of the order of rows and columns mitigates the salience of the dominance solution actions while precluding subjects from coordinating on outcomes that are Pareto superior with respect to the dominance solution.

Diagonal games constitute a valuable toolkit for generating games with a clear step-reasoning structure. While other games were proposed to that effect — e.g. the 11-20 game (Arad and Rubinstein 2012) or ring games (Kneeland 2015) —, diagonal games are unique in their ability to enable meaningful comparative statics for different models with respect to game complexity. Diagonal games not only allow for a varying number of steps to reach the dominance-solution and a varying number of actions, they are also easy to generate and facilitate cross-game comparisons. With this benchmark class of games in hand, I am hopeful that the study of complexity and cognitive costs in strategic settings becomes more amenable to both formal analysis and experimental research.

[1]Some examples are Stahl and Wilson (1994); Stahl and Wilson (1995), Costa-Gomes et al. (2001), Costa-Gomes and Weizsäcker (2008) and Rey-Biel (2009).

Outline

In Section 1, I introduce the class of diagonal games and explore their properties. In Section 2, I focus on the theoretical predictions implied by existing models. I examine the two additional variants of diagonal games mentioned earlier in Section 3. I discuss experimental implementations in Section 4, where I also contrast diagonal games with other classes of games. Finally, I present some concluding remarks in Section 5. Proofs can be found in the Appendix.

1. Definitions and Properties

I will start by defining diagonal games:

Definition 1 (Diagonal Game)**.** A diagonal game, given by $\Gamma = \langle v_1, v_2, N\rangle$, corresponds to a finite normal-form two-player game $\langle I, A, u\rangle$ where $I = \{1,2\}$ denotes the set of players with general elements i and $j \neq i$; $A = \{1,\dots,N\}$ denotes the set of actions available to each player; and $u = (u_1, u_2)$, where $u_i : A^2 \to \mathbb{R}$ denotes player i's payoff function, given by $u_i(a_i, a_j) = v_i(N - a_i + a_j)$, for some function $v_i : \mathbb{Z} \to \mathbb{R}$.

The reason why these games are called diagonal games is that players' payoffs are constant along parallels to the main diagonal as illustrated in Figure 1.

		Player j		
		1	2	3
	1	$v_i(3)$	$v_i(4)$	$v_i(5)$
Player i	2	$v_i(2)$	$v_i(3)$	$v_i(4)$
	3	$v_i(1)$	$v_i(2)$	$v_i(3)$

Figure 1. **Player i's Payoffs**

Before proceeding, let us recall the concept of dominance-solvability. Let $A_i^0 \equiv A_i$ and, for any $t = 1,2,\ldots$ and $i \in I$, define

$$A_i^t := \left\{ a_i \in A_i^{t-1} \,\middle|\, \nexists \sigma_i \in \Delta(A_i) \text{ such that } \int_{a_i' \in A} u_i(a_i', a_j)\, d\sigma_i(a_i') > u_i(a_i, a_j),\ \forall a_j \in A_j^{t-1} \right\}. \tag{1}$$

The set A_i^1 characterizes the set of player i's actions that are not strictly dominated. Then, iteratively, the set A_i^t includes all of player i's actions that are not strictly dominated considering that player j will only choose actions in A_j^{t-1}, that is, actions that were not deleted up to the t-th iteration. The game is then said to be dominance-solvable when, for all players $i \in I$, $A_i^\infty = \bigcap_{t \in \mathbb{N}} A_i^t$ is a singleton, that is, when only one action survives the iterated deletion process.

In order to render diagonal games dominance-solvable it will be sufficient to consider the following assumption:

Assumption 1 (Single-peaked)**.** For $i = 1,2$, v_i is single-peaked, with $\arg\max_{n \in \mathbb{Z}} v_1(n) = \arg\max_{n \in \mathbb{Z}} v_2(n) > N$, and is strictly increasing up to its maximum.

To explain the logic as to why Assumption 1 will deliver us dominance-solvability, let us define the following implied parameter:

$$h := \arg\max_{n \in \mathbb{Z}} v_i(n) - N$$

Let us consider the game in Figure 1 and suppose $h = 1$. This implies that $v_i, i = 1,2$, peaks at 4. Then, as v_i is strictly increasing on $\{1,\ldots,N+h\}$, action 3 is strictly dominated by action 2: $u_i(3,a_j) = v_i(a_j) < v_i(a_j+1) = u_i(2,a_j)$, for $a_j = 1,2,3$. Upon deleting action 2 for both players, it is then straightforward to check that action 2 is strictly dominated by action 1 and thus the dominance solution is (1,1). It is worth noting that, insofar as the functions v_i have the same maximizer, the deletion order is the same for both players, even if the functions v_i are not. This next proposition shows not only how the logic extends to arbitrary N and h, but also that these parameters pin-down the number of steps of iterated deletion of dominated actions needed to reach the dominance-solution. It further shows that diagonal games are not only dominance-solvable, but also supermodularity.[2]

Proposition 1. Let Γ be a diagonal game and suppose Assumption 1 holds. Then Γ is a supermodular and dominance-solvable game. Moreover,

(i) if $h \geq N$, then Γ is solvable in one step of iterated deletion of dominated actions, that is, both players have a dominant action; and

(ii) if $h \leq N-1$, then Γ is solvable in exactly $T := \lceil \frac{N-1}{h} \rceil$ steps of iterated deletion of dominated actions and, in each $t < T$, h actions are deleted.

Diagonal games also provide an intuitive way to rank the actions.

Definition 2 (Strategic Ranking of Actions)**.** For $i = 1,2$, let $\rhd_{i,A}$ be such that $\forall a_i, a_i' \in A$, $a_i' \rhd_{i,A}$ if and only if $\exists t \in \mathbb{N} : \{a_i', a_i\} \subseteq A_i^t$ and $u_i(a_i', a_j) > u_i(a_i, a_j)$, $\forall a_j \in A_j^t$, where A_i^t is defined as in (1).

This definition has $\rhd_{i,A}$ ranking actions by iterated dominance considering only the surviving actions of the opponent. While dominance-solvability immediately implies that $\rhd_{i,A}$ is a partial order over the set of actions, the structure of diagonal games under the above assumption delivers a sharper characterization.

Proposition 2. Let Γ be a diagonal game and suppose Assumption 1 holds. Then, $\rhd_A \equiv \rhd_{1,A} = \rhd_{2,A}$ and $\rhd_A$ is a linear order over A such that, $\forall a_i, a_i' \in A$, $a_i' \rhd_A a_i$ if and only if $a_i' \leq a_i$.

Proposition 2 states that, under the above assumption, the ranking of actions is the same for both players, ranks every action and coincides with the (reversed) natural ordering of $\mathbb{N}$. Simply put, under Assumption 1, actions corresponding to smaller numbers are "closer" to the dominance solution in a well-defined sense. Note that it is not the case that a similar result holds for arbitrary dominance-solvable games, in particular, it is not generally true that iterated dominance would deliver a linear order. For instance, one of

[2]This paper follows Milgrom and Shannon's (1994) definition of a supermodular game, relaxing conditions (A3) and (A4) in Milgrom and Roberts (1990), replacing supermodularity for quasisupermodularity (A3) and increasing differences for the single-crossing property (A4).

the players could have a dominant action, but, among the remaining actions, there may not be any dominant or dominated action, in which case the remaining actions would not be comparable according to $\rhd_A$. In light of Proposition 2 and for the sake of notational simplicity I will henceforth identify $\rhd_A$ with the reversed order on natural numbers.

Throughout the next section I will rely on a specific way to rank games according.

Definition 3 (Complexity Ranking). Let $\Gamma = \langle v_1, v_2, N\rangle$ and $\Gamma' = \langle v_1', v_2', N'\rangle$ be two diagonal games satisfying Assumption 1 and let $N+h$ and $N'+h'$ denote the maximizers of v_i and v_i', $i = 1,2$, respectively. I will say that diagonal game Γ' is **more complex than** Γ, denoted by $\Gamma' \rhd_C \Gamma$ if $N' - N \geq h - h' \geq 0$ and if, for $i = 1,2$,

(i) $v_i'(n) = v_i(n + (N+h) - (N'+h'))$, for any n; or

(ii) $v_i'(n) = v_i(n+(N-N'))$ for $n \leq N'+h'$ and $v_i'(n) = v_i(n+(N+h)-(N'+h'))$ for $n > N'+h'$.

Intuitively, while N, the number of actions available to the players, naturally relates to a notion of size complexity, h captures a notion of strategic complexity. Then, increasing the number of actions makes a game more complex, insofar as the number of actions eliminated at each step of iterated deletion of dominated actions does not increase. Similarly, keeping the action set size fixed, having fewer actions deleted at each step — a smaller h — increases the number of steps necessary to reach the dominance solution. Conditions (i) and (ii) in the definition of the proposed complexity ranking provide cardinal conditions that will simplify the analysis with respect to different solution concepts. On the other hand, these conditions also put forth a practical way to take a specific diagonal game and transform the functions (v_1, v_2) and on the size of the action set in order to generate another diagonal game which is comparable to the original one with regard to the complexity ranking $\rhd_C$.

In order to exemplify the conditions (i) and (ii), consider the following (symmetric) diagonal game $\Gamma = \langle v, v, 4\rangle$, with $v(n) = \mathbf{1}_{n\leq 6} \cdot 10 \cdot n$, where $\mathbf{1}_{(\cdot)}$ represents the indicator function. This game has players' payoffs as represented in panel 2a of Figure 2. This specification of payoffs implies that $h = 2$ as we have that $\arg\max_{n\in\mathbb{Z}} v(n) = 6$. Consequently, by Proposition 1, we know that this game is dominance-solvable in $\lceil \frac{4-1}{2} \rceil = 2$ steps.

		Player j			
		1	2	3	4
Player i	1	40	50	60	0
	2	30	40	50	60
	3	20	30	40	50
	4	10	20	30	40

(a) Γ

		Player j				
		1	2	3	4	5
Player i	1	40	50	60	0	0
	2	30	40	50	60	0
	3	20	30	40	50	60
	4	10	20	30	40	50
	5	0	10	20	30	40

(b) Γ^1

		Player j			
		1	2	3	4
Player i	1	50	60	0	0
	2	40	50	60	0
	3	30	40	50	60
	4	20	30	40	50

(c) Γ^2

		Player j			
		1	2	3	4
Player i	1	40	50	0	0
	2	30	40	50	0
	3	20	30	40	50
	4	10	20	30	40

(d) Γ^3

Figure 2. **Complexity-Ranked Diagonal Games; Player i's Payoffs**

Suppose we know want to generate games that are ranked as more complex than Γ according to $\rhd_C$. In particular, we want to go from $(N,h) = (4,2)$ to games with more actions $(5,2)$ and, differently, to games where fewer actions are deleted at each step, $(4,1)$, but keeping a comparable payoff structure. Game Γ^1 in panel 2b of Figure 2 achieves our first goal. It retains a similar payoff structure and adds a dominated action by letting $N^1 = 5$ and $h^1 = 2$ and defining $v^1(n) = v(n+(4-5))$, which, in this case, satisfies both conditions (i) and (ii). Moreover, it now requires $\lceil \frac{5-1}{2} \rceil = 3$ steps to reach the dominance solution. Games Γ^2 and Γ^3 in panels 2c and 2d, respectively, accomplish our second goal and showcase the difference between conditions (i) and (ii). To obtain Γ^2 from Γ one can simply set $v^2(n) = v(n+(N+h)-(N^2+h^2)) = v(n+1)$ — satisfying condition (i) — whereas Γ^3 relies on $v^3(n) = v(n+(N-N^3)) = v(n)$ for $n \leq N^3+h^3 = 5$, and $v^3(n) = v(n+(N+h)-(N^2+h^2)) = v(n+1)$, for $n > 5$, which satisfies condition (ii). Both these games have $(N^2,h^2) = (N^3,h^3) = (4,1)$ and are therefore dominance-solvable in $\lceil \frac{4-1}{1} \rceil = 3$ steps.

The complexity ranking $\rhd_C$ relates to the notion of time complexity in computer science, which describes the number of steps required for an algorithm to terminate. Other rankings that capture some notion of complexity are possible. For instance, one could rank diagonal games by the number of steps of deletion of dominated actions that is needed to reach the dominance solution. While this is an appealing proposal that turns out to not provide the necessary restrictions to characterize predictions for models that make use of the cardinal differences and not just ordinal ranking of payoffs. In contrast, as shown in the next section, different solution concepts yield meaningful comparative statics across diagonal games ranked according to $\rhd_C$, supporting its theoretical appeal.

2. Theoretical Predictions and Comparative Statics

In this section, I will examine the predictions made by different models. As the main purpose of this new class of games is to provide a valuable benchmark to study complexity across games, it is particularly relevant to understand whether it delivers testable predictions for models that focus on cognitive limitations to explain gameplay.

In order to obtain sharp results, throughout this section I will consider the following restriction on the functions characterizing the diagonal game:

Assumption 2 (Cliff). For any $m > \arg\max_{n \in \mathbb{Z}} v_i(n)$, $v_i(m) = \min_{n \in \mathbb{N}} v_i(n)$.

Assumptions 1 and 2 together imply that v_i is increasing up to the peak and then drops to a constant, resembling a cliff.

2.1. Step-Reasoning

I will first consider one of the most well-known models of reasoning in strategic settings, level-k. It posits that players' choices is characterized by their type, $k = 0, 1, 2, \ldots$, which captures their beliefs about their opponents' gameplay. In this model, level k players choose the action that best-responds to level $k-1$ players and, following the literature on this and related models,[3] level 0 is taken to choose uniformly at random. This model then captures cognitive limitations of the players through the number of steps in their reasoning is now used for a number of applications, from mechanism design (de Clippel et al. 2018) to new Keynesian models (Farhi and Werning 2019).

The first proposition in this section shows how level-k actions are uniquely and robustly identified.

[3] See, e.g., Stahl and Wilson (1994; 1995), Costa-Gomes et al. (2001) and Camerer et al. (2004).

Proposition 3 (Level-k Ranking of Actions)**.** Let Γ be a diagonal game and suppose Assumptions 1 and 2 hold. Then $\forall k \in \mathbb{N}$, a level-k player will choose the action $a_i^k = \max\{1, N - k \cdot h\}$.

I want to highlight that Proposition 3 as well as the other results only depend on v_1 and v_2 ordinally as any strictly increasing transformation does not affect the implied parameter h. A simple consequence of this fact is that, if $u_i(a_i, a_j)$ describes players' monetary payoffs instead of their utility as is standard in experiments, insofar as players are expected utility maximizers, the identification of level-k actions in the above proposition still holds regardless of their risk attitudes.[4] This is due to the fact that payoffs are constant along the parallels to the main diagonal together with the fact that v_i drops after achieving its maximum.

There are several models of step-reasoning related to level-k. One such example is that of dominance-k (Costa-Gomes et al. 2001). According to this model, a dominance-k player performs $k \in \mathbb{N}$ rounds of deletion of dominated actions and then best-responds to a uniform distribution over the surviving actions of the opponent, i.e. a uniform distribution over A_j^k. A similar result to that in Proposition 3 holds, identifying the actions chosen by such players.

Proposition 4 (Dominance-k Ranking of Actions)**.** Let Γ be a diagonal game and suppose Assumptions 1 and 2 hold. Then $\forall k \in \mathbb{N}$, a dominance-k player will choose the action $a_i^k = \max\{1, N - (k+1) \cdot h\}$.

A second related model is most common in experimental applications ans involves allowing payoff-dependent noise in level-k players best-response to their beliefs (e.g. Costa-Gomes and Weizsäcker 2008). Such a model can be thought of as a mixture between quantal response equilibrium, where players best-respond imperfectly to correct beliefs about their opponents' gameplay, and level-k models, where players best-respond perfectly to incorrect beliefs. A generalization of quantal response, rank-dependent choice, was recently introduced by Goeree et al. (2019), which posits only that the probability that a

[4] This stands in contrast to the 11-20 game (Arad and Rubinstein 2012) and its dominance-solvable variant (Alaoui et al. 2020), where the action chosen by level-1 players would depend on risk attitudes if level-0 is assumed to choosing uniformly at random and payoffs corresponded to monetary payoffs and not utility. It is important to note, though, that the identification would go through with the more heuristic assumption that level-0 players choose a specific action, 20, that is particularly salient in that game.

given action being chosen by a player is increasing in its expected payoff, according to the players' beliefs. This avoids any parametric assumptions on the payoff-dependent noise. In this spirit, let us define a general model of *level-k with payoff-dependent mistakes* as one where players' beliefs are given by the level-k model — i.e. level-k players believe their opponents will choose the level-$(k-1)$ action with probability one — and choose any given action with a probability that is increasing the in expected payoff given such beliefs. A corollary to Proposition 3 provides the following testable implications in diagonal games:

Corollary 1. Let Γ be a diagonal game and suppose Assumptions 1 and 2 hold. Then $\forall k \in \mathbb{N}$, a level-k player with payoff-dependent mistakes,

(i) chooses action $\max\{1, N-k\cdot h\}+n$ with weakly greater probability than action $\max\{1, N-k\cdot h\}+n+1$, for any $n = 0, \ldots, \min\{N-2, k\cdot h-1\}$;

(ii) all actions $a_i < N-k\cdot h$ are chosen with the same probability; and

(iii) action N is chosen with weakly greater probability than any action $a_i < N-k\cdot h$.

The intuition behind Corollary 1 is simple. If player i believes the opponent is choosing action n with probability 1, then the expected payoffs are highest for action $n-h$, achieving $v_i(N+h) = \max_{a_i,a_j} u_i(a_i,a_j)$ and decrease when player i chooses actions corresponding to larger numbers. Given Assumption 2, $u_i(a_i,n) = \min_{n'} v_i(n')$ for any $a_i < n-h$, which then implies that choosing numbers smaller than $n-h$ results in the lowest possible payoff. Then, as rank-dependent choice determines that the probability with which player i chooses any given action is increasing in its payoff, the result follows.[5]

While the above results characterize gameplay within a single diagonal game, the next result delivers cross-game comparative statics with respect to the complexity ranking of diagonal games characterized earlier.

Corollary 2. Within the class of diagonal games that satisfy Assumptions 1 and 2 and are ranked according to $\rhd_C$,

[5] Rank-dependent choice equilibrium (and, consequently, quantal response equilibrium) also have one testable implication, albeit weaker. For any diagonal game satisfying Assumption 1, in any such equilibrium and for any of the two players, action $N-h+m$ is chosen with weakly great probability than action $N-h+m+1$, for $m = 0, \ldots, h-1$. This is a robust prediction that is trivially implied by the fact that action $N-h+m$ strictly dominates action $N-h+m+1$, for any $m = 0, \ldots, h-1$.

(i) the action played by a level-k player is decreasing in $\triangleright_C$ with respect to $\triangleright_A$;

(ii) the action played by a dominance-k player is decreasing in $\triangleright_C$ with respect to $\triangleright_A$; and

(iii) the modal action played by a level-k player with payoff-dependent mistakes is decreasing in $\triangleright_C$ with respect to $\triangleright_A$.

Note that it is not sufficient to have arbitrary dominance-solvable games that are ranked in the number of actions and the number of steps needed to reach the dominance solution in order to obtain claims (ii) and (iii) in Corollary 2. For instance, it is easy to construct counterexamples even with dominance-solvable games that have the same number of actions, require the same number of steps of deletion to reach the dominance solution and have the same actions deleted at each step.

2.2. Endogenous Depth of Reasoning

A recent development in models relating cognitive abilities and gameplay proposes that the players' cognitive effort is endogenous to the strategic setting. One such example is the endogenous depth of reasoning model by Alaoui and Penta (2016), which provides a way to endogenize level-k reasoning as resulting from a cost-benefit analysis.

Let us introduce the main elements of this model, adapted to the specific context of diagonal games. First, the model posits that each player i has a cost of (step) reasoning, $c_i : \mathbb{N} \to \mathbb{R}_+$. While the model can allow for players to have beliefs about their opponent's cost of reasoning as well as second-order beliefs, in this paper I will consider the simpler version of the model, with fewer degrees of freedom for the analyst, by assuming that costs of reasoning are common knowledge. Second, players follow a path of reasoning, resulting from a cost-benefit analysis, denoted by $R_i = (a^i_{i,k}, a^i_{j,k-1})_{k\geq 1}$, where $a^i_{j,k-1}$ denotes the (potentially mixed) action that, after k steps of reasoning, player i believes their opponent will take. In particular, the model determines that $a^i_{i,k}$ ($a^i_{j,k}$) is a best-response to $a^i_{j,k-1}$ ($a^i_{i,k-1}$), when there is a unique best-response; when otherwise, $a^i_{i,k}$ ($a^i_{j,k}$) corresponds to a uniform distribution over the set of best-responses to $a^i_{j,k-1}$ ($a^i_{i,k-1}$). The actions ($a^i_{i,0}, a^0_{j,0}$), called player i's anchor, are exogenous given, with $a^i_{j,0}$ representing player i's starting belief about their opponent's choices and $a^i_{i,0}$ denoting the action that player i

chooses when not taking any step of reasoning. Given the fact that diagonal games imply symmetric deletion paths, I will assume that $a^i_{i,0} = a^i_{j,0} \equiv a^i_0$ and that these correspond to either an action (non-mixed) or the uniform distribution over actions.[6]

Players' decision to stop reasoning follows a cost-benefit calculation and, for this purpose, a notion of the value of reasoning is needed. In this paper I will adopt the "maximum gain" representation of the value of reasoning,[7] which is given by

$$V_i(k) := \max_{a_i, a_j} u_i(a_i, a_j) - u_i(a^i_{i,k-1}, a_j). \tag{2}$$

Finally, the model assumes that players follow a myopic stopping rule, deciding to stop reasoning after $\hat{k}_i$ steps, where

$$\hat{k}_i := \min\{k \in \mathbb{N}_0 \mid c_i(k) \leq V_i(k) \text{ and } c_i(k+1) > V_i(k)\}.$$

The term $\hat{k}_i$ is dubbed player i's cognitive boundary and both players are aware of the other's cognitive boundary, taking that into account in their reasoning process. I will denote by a^*_i the action that player i ultimately chooses, that is,

$$a^*_i \equiv a^i_{i,\hat{k}_i} := \arg\max_{a_i \in A_i} u_i(a_i, a_{j,\hat{k}^i_j}), \quad \hat{k}^i_j := \min\{\hat{k}_i - 1, \hat{k}_j\}.$$

The first observation regarding the predictions of the endogenous depth of reasoning model in diagonal games is that the actions in any path of reasoning are increasing with respect to the strategic ranking $\rhd_A$. Formally,

Lemma 1. Let Γ be a diagonal game and suppose Assumption 1 holds. Then, for any $k \in \mathbb{N}_0$, $a_{i,k+1} \rhd_A a_{i,k}$ and, in particular, $a_{i,k} = \max\{1, a^i_0 - k \cdot h\}$.

Lemma 1 can be interpreted as stating that, in this class of games, reasoning more always brings the players "closer" to the dominance solution actions, regardless of the specification of the cost and value of reasoning. It is easy to explain why this is the case. Given that,

[6] I discuss the effects of relaxing this assumption in the Appendix Appendix A, namely the fact that choices cease to be monotone. This is a feature of endogenous depth of reasoning and not of diagonal games. For instance, this non-monotonicity would also occur in dominance-solvable variants of the 11-20 game, implying that this assumption — made by the authors — is in fact necessary for some of the results in Alaoui and Penta (2016).

[7] See Alaoui and Penta (2020, Theorem 5) for an axiomatic characterization.

for any diagonal game satisfying Assumption 1 and for any action of the opponent a_j, the player's best-response is given by $\max\{1, a_j - h\}$, the result is immediately deducible by induction.

A second lemma that will prove useful identifies the value of reasoning under the maximum gain specification in (2). For notational convenience, let $\underline{v}_i := \arg\min_{n\in\mathbb{N}} v_i(n)$, which is well-defined under Assumptions 1 and 2.

Lemma 2. Let Γ be a diagonal game and suppose Assumptions 1 and 2 hold. Then, $V_i(1) \leq V_i(k) = v_i(N+h) - \underline{v}_i$, for $k \geq 2$, holding with equality when $a_0^i < N_h$.

Lemmata 1 and 2 simplify the identification of the player's cost of reasoning from the data by requiring the only manipulation of $v_i(N+h)$ and $\underline{v}_i$ and the observation of the chosen action. Moreover, they also provide clear comparative statics with respect to incentives:

Proposition 5. Let Γ be a diagonal game and suppose Assumptions 1 and 2 hold. Then, if either (i) c_i decreases, (ii) $v_i(N+h)$ increases, or (iii) $\underline{v}_i$ decreases for one of the players $i \in \{1,2\}$, then (a_1^*, a_2^*) increase with respect to $\rhd_A$. Moreover, if $\hat{k}_i \geq 1$, then increasing $v_i(n)$, $n = 1, \ldots, N+h-1$ does not affect (a_1^*, a_2^*).

Proposition 5 states that when the cost of reasoning is lower or the difference $v_i(N+h) - \underline{v}_i$ increases for any of the players, both players choose actions that are (weakly) "closer" to the dominance solution, that is, they choose lower numbers.

Similarly to exogenous step-reasoning models, when players' depth of reasoning is endogenous, more complex diagonal games will induce the actions ranked lower according to the strategic ranking $\rhd_A$. The next proposition formalizes this claim:

Proposition 6. Let $\Gamma, \tilde{\Gamma}$ be diagonal games satisfying Assumptions 1 and 2 such that $\tilde{\Gamma} \rhd_C \Gamma$ and let the corresponding player i's anchor, costs, cognitive boundary and action chosen be denoted by $(a_0^i, c_i, \hat{k}_i, a_i^*)$ and $(\tilde{a}_0^i, \tilde{c}_i, \tilde{k}_i, \tilde{a}_i^*)$. If $\hat{k}_i \geq 1$, $a_0^i \rhd_A \tilde{a}_0^i$ and $c_i \leq \tilde{c}_i$ for $i = 1,2$, then $a_i^* \rhd_A \tilde{a}_i^*$ for $i = 1,2$.

Proposition 6 implies that if (i) the anchors are kept fixed at a given number or if they correspond to, say, a given level-k action in each of the games, and (ii) the costs of reasoning are not lower for the more complex game, then if the players take at least one step of reasoning in the simpler game, they will ultimately choose actions which are "farther away" from the dominance solution (i.e. larger numbers) in the more complex game.

2.3. Sampling Equilibrium

Diagonal games also provide testable implications for models that do not rely on step reasoning. In this section, I discuss the predictions and comparative statics of a solution concept that posits a different mechanism for how players form beliefs about their opponents' gameplay: a variant of Osborne and Rubinstein's (2003) sampling equilibrium.

Suppose that, in the context of a diagonal game, player i starts with a prior μ_i over their opponents' distribution of actions, $\mu_i \in \Delta(\Delta(A))$. That is, player i is unsure about which mixed strategy in $\Delta(A)$ player j is going to choose. Then, player i is able to observe k independent draws from their opponent's mixed strategy, $a_j^k = (a_{j,1},\dots,a_{j,k})$ with $a_{j,\ell} \overset{\text{iid}}{\sim} \sigma_j$, $\ell = 1,\dots,k$. Following such draw, the player updates beliefs according to Bayes' rule, forming the posterior belief $\mu_i \mid a_j^k$, and best-responds to that posterior belief, $\sigma_i^*(\mu_i \mid a_j^k) \in \arg\max_{\sigma_i' \in \Delta(A)} \int_{\sigma_j' \in \Delta(A)} u_i(\sigma_i', \sigma_j') d(\mu_i \mid a_j^k)(\sigma_j')$.

While it corresponds to a special case of the original definition in Osborne and Rubinstein (2003), I will call a pair of mixed strategies $(\sigma_1, \sigma_2) \in \Delta(A) \times \Delta(A)$ a *k-sampling equilibrium* if, for $i = 1,2$, $\sigma_i = \mathbb{E}[\sigma_i^*(\mu_i \mid a_j^k)]$ for some selection of best-responses to such posterior beliefs $\sigma_i^*(\mu_i \mid a_j^k)$. This model effectively corresponds to Salant and Cherry's (2020) sampling equilibrium with statistical inference — originally defined only for a specific class of games — adjusted to the class of diagonal games and is also closely related to sequential sampling equilibrium (Gonçalves 2020), which endogenizes the information that players acquire by rendering the sampling procedure costly and sequential. Being a special case of Osborne and Rubinstein's (2003) model, it follows immediately that a sampling equilibrium always exists.[8]

This sampling procedure can be interpreted as the players being able to access a number of observations from past play. Alternatively, one can think of this sampling procedure as a way to model the players' internal deliberation, where sampling refers to noise in the deliberation and the sample size, k, constituting a proxy for individuals noisy reasoning. This interpretation is especially appealing as when the number of samples grows, by the

[8]The proof relies on a simple application of Brouwer's fixed-point theorem.

strong law of large numbers, sampling equilibria converge to Nash equilibrium whenever the priors are well-behaved, namely, if the priors are absolutely continuous with respect to the Lebesgue measure.

For the sake of tractability, I will impose two assumptions throughout the remainder of this section. The first is to constrain the priors to be uniform, where all distributions of actions of the opponent are ex-ante equally likely. While an intuitive and defensible assumption, it is imposed mainly for tractability as this implies the prior corresponds to a Dirichlet distributions with parameters $\alpha_0 = (1)^N$, making belief updating extremely tractable as well as delivering a straightforward connection between the posterior mean and the samples. The second assumption is a further restriction on the payoff functions:

Assumption 3 (Linearity)**.** For any $n < N + h$, $v_i(n) = v_i(n-1) + \delta_i$, $i = 1,2$.

Assumption 3 together with Assumptions 1 and 2, implies that we can write v_i as

$$v_i(n) = \mathbf{1}_{n \leq N+h} \cdot (\beta_i + n \cdot \delta_i) + \mathbf{1}_{n > N+h} \cdot \underline{v}_i, \tag{3}$$

where $\beta_i + \delta_i \geq \underline{v}_i$, $\delta_i > 0$, $N \in \mathbb{N}$, and $h \geq 1$. This leads to a simple way of generating diagonal games, which are then characterized by a handful of parameters. Moreover, players' payoffs then take a simple form:

$$u_i(a_i, a_j) = \mathbf{1}_{a_j - a_i \leq h} \cdot (\beta_i + (N - a_i + a_j) \cdot \delta_i) + \mathbf{1}_{n > N+h} \cdot \underline{v}_i.$$

Finally, let us provide a way to rank k-sampling equilibria according to the strategic ranking of actions $\rhd_A$. I will say that, for any two profiles of action distributions, $\sigma, \sigma' \in \Delta(A)^2$, $\sigma \rhd_{A,FOSD} \sigma'$ if, for $i = 1,2$, σ_i first-order stochastically dominates σ'_i with respect to $\rhd_A$. That is, if $\sum_{n=0}^{m} \sigma_i(m) \geq \sum_{n=0}^{m} \sigma'_i(m)$ for any $m \geq 0$, with σ assigning greater probability to actions that are ranked lower according to $\rhd_A$. Furthermore, I will call a k-sampling equilibrium σ the $\rhd_{A,FOSD}$-largest equilibrium if it $\rhd_{A,FOSD}$-dominates any other k-sampling equilibrium. We then have the following characterization of k-sampling equilibria:

Proposition 7. Let Γ be a diagonal game and suppose Assumptions 1-3 hold. For any $h, k \geq 1$, any k-sampling equilibrium satisfies $\sigma_i(n) = 0$ $\forall n \in \{1, \ldots, N - \max\{k, h\} - 2\} \cup \{N - h + 1, \ldots, N\}$. Moreover,

(i) when $\left\lceil \frac{\beta_i+\delta_i-\underline{v}_i}{\beta_i} \right\rceil = M$, for $i = 1,2$, then a $\rhd_{A,FOSD}$-largest k-sampling equilibrium exists and is given by $\sigma_i(n^*) = 1$, where $n^* = \min\{N-h, N+M-k-1\}$; and

(ii) when $h \geq k \geq 1$, any k-sampling equilibrium satisfies $\sigma_i(N-h-1)+\sigma_i(N-h) = 1$ and, if $h > k$ or $\beta_i + \delta_i > \underline{v}_i$ for $i = 1,2$, then the unique k-sampling equilibrium satisfies $\sigma_i(N-h) = 1$.

The above proposition states that for an equilibrium to place strictly positive probability on actions ranked higher according to $\rhd_A$ the number of samples needs to be large enough. Claim (i) refines this observation and shows that when payoffs are close to being symmetric, there is a $\rhd_{A,FOSD}$-largest k-sampling equilibrium which increases with respect to $\rhd_{A,FOSD}$ in the number of samples. Claim (ii) uncovers an interesting relation between the number of samples k and the parameter h: whenever the former is strictly smaller than the latter, then in equilibrium both players choose the level-1 action with probability 1.

An immediate corollary relating complexity and k-sampling equilibrium follows:

Corollary 3. Let $\Gamma, \tilde{\Gamma}$ be diagonal games satisfying Assumptions 1-3 such that $\tilde{\Gamma} \rhd_C \Gamma$. If $\left\lceil \frac{\beta_i+\delta_i-\underline{v}_i}{\beta_i} \right\rceil = M$ for $i = 1,2$, or $h \geq k$, then the $\rhd_{A,FOSD}$-largest equilibrium in Γ $\rhd_{A,FOSD}$ dominates that in $\tilde{\Gamma}$.

3. Variants

I consider two variants of the class of diagonal games. One constitutes an extension where, instead of a finite set of actions, players can choose from a continuum of actions. There, the set of alternatives is characterized by a closed interval on the real line, similarly to beauty contest games, and I show that many of the theoretical predictions characterized above still hold. The other variant is a close relative of diagonal games whereby changing a single payoff for each player, out of N^2, changes the strategic nature of the game to having all Nash equilibria in mixed strategies and, furthermore, when the game is symmetric, a unique symmetric Nash equilibrium. This second variant enables the comparison of game-play across similar games but where Nash equilibrium makes very different predictions.

3.1. Interval Diagonal Games

Similar to diagonal games, an *interval diagonal game* corresponds to a normal-form two player game $\langle I, A, u\rangle$ and is characterized by $\langle v_1, v_2, N\rangle$, where $I = \{1,2\}$, $A = [1,N]$, $v_i : \mathbb{R} \to \mathbb{R}$ and $u = (u_1, u_2)$ with $u_i : A^2 \to \mathbb{R}$ and $u_i(a_i, a_j) = v_i(N - a_i + a_j)$. I will consider the following analogues of Assumptions 1 and 2:

Assumption 1-I. For $i = 1,2$, v_i is single-peaked, with $\arg\max_{n\in\mathbb{R}} v_1(n) = \arg\max_{n\in\mathbb{R}} v_2(n) > N$, and is strictly increasing up to its maximum.

Assumption 2-I. For any $m > \arg\max_{n\in\mathbb{R}} v_i(n)$, $v_i(m) = \min_{n\geq 1} v_i(n)$.

It is straightforward to verify that the previous results referring to the level-k, the dominance-k and the endogenous depth of reasoning models also hold for interval diagonal games, relying on Assumptions 1-I and 2-I instead of 1 and 2.[9] If one is willing to assume that level-k with payoff-dependent mistakes induces an absolutely continuous distribution over actions and rank-dependence refers to the density instead of a probability mass function, analogue implications to the ones derived earlier for this model also remain valid for interval diagonal games.[10]

3.2. Mixed Diagonal Games

A simple transformation of Assumptions 1 and 2 yields very similar games but, as we shall see, inducing a unique Nash equilibrium in non-degenerate mixed strategies.

Assumption 1-M. For $i = 1,2$, (i) v_i is single-peaked on $\{n \in \mathbb{Z} \mid n \geq 2\}$ with $\arg\max_{n\geq 2} v_1(n) = \arg\max_{n\geq 2} v_2(n) > N$, and is strictly increasing up to its maximum on $n \geq 2$, and (ii) $v_i(1) > v_i(N)$.

Assumption 2-M. For any $m > \arg\max_{n\geq 2} v_i(n)$, $v_i(m) = \min_{n\geq 2} v_i(n)$.

Assumption 1-M breaks the dominance-solvability delivered by Assumption 1 as now $u_i(N,1) = v_i(1) > v_i(N) = u_i(1,1)$, making action N a strict best-response to action 1. Assumption 2-M is then just the analogue of Assumption 2. It should be straightforward that

[9] That is, Propositions 1-6, parts (i) and (ii) of Corollary 2 and Lemmata 1-2.
[10] Corollary 1 and part (iii) of Corollary 2.

in any diagonal game satisfying Assumption 1-M, all Nash equilibria are in non-degenerate mixed strategies.

To state the main result in this section, I will need to introduce some additional notation. In line with Assumption 1-M let $h' := \arg\max_{n\geq 2} v_i(n) - N$ and $T' := \lceil (N-1)/h' \rceil$. Furthermore, for any $k \in \mathbb{N}$, let $k' : k \mapsto k'$ such that

$$k' := \begin{cases} k \bmod_1 T' + 1 & \text{if } k \bmod_1 T' + 1 < T' + 1 \\ 0 & \text{if otherwise} \end{cases},$$

where $a \bmod_m n$ denotes the modulo-n operator with an offset of m, that is, $a \bmod_m n := a - n \cdot \lfloor (a-m)/n \rfloor$. We then have the following result:

Proposition 8. Let Γ be a diagonal game and suppose that Assumptions 1-M and 2-M hold.

(i) An action a_i is chosen by level-k players with $k \geq 1$ if and only if $a_i \in \{\max\{1, N - n \cdot h'\}, n \in \mathbb{N}\}$. If $v_i(1) < v_i(N + h')$, then $\forall k \in \mathbb{N}$, a level-k player will choose the action $a_i^k = \max\{1, N - k' \cdot h'\}$.

(ii) If the game is symmetric ($v_1 = v_2$), there is a unique symmetric Nash equilibrium σ. Furthermore, σ_i has support on the set of actions chosen by level-k players, $k \geq 1$.

Proposition 8 provides a simple way to transform a symmetric diagonal game $\langle v, v, N \rangle$ satisfying the restriction given by Assumptions 1 and 2 — hence, by Proposition 1, dominance-solvable — into a game with a unique symmetric Nash equilibrium in non-degenerate mixed strategies. Moreover, this is done while retaining identification of step-reasoning structure — up to cycling, of course — and by making minimal transformations on the players' payoffs, changing the payoff associated to one single outcome out of N^2, that is, $u_i(N,1) = v(N+h)$ to $u_i(N,1) = v(N+h)$.

4. Related Games and Experimental Implementation

The previous sections illustrated the usefulness of diagonal games as a benchmark for deriving testable implications for a variety of models, from step-reasoning models — level-k, dominance-k, endogenous depth of reasoning — to Nash equilibrium and k-sampling equilibrium. In this section, I will discuss some features of this novel class of games that make them particularly suitable for experimental applications. Before that, though, I review other existing games that share some of the properties of diagonal games and examine their differences and similarities.

4.1. Related Games

The three classes of games that are the most closely related to diagonal games are beauty contest games, the 11-20 game, and ring games. The interval variant of diagonal games is easily comparable to the beauty contest game (Nagel 1995), where players have to pick a number from 0 to 100 and have to guess a fraction $p \in (0,1)$ of the resulting average. Only the winner gets a prize and ties are broken at random or the prize is divided.

The three main differences between beauty contest games and diagonal games. First, diagonal games have a natural translation from interval action spaces to finite ones (and back, via e.g. linear interpolation of v_i). Second, the flexibility in the payoff function allows for easier manipulations, which can be particularly convenient to derive and test comparative statics.[11] Finally, in contrast to diagonal games, beauty contest games cannot be ranked according to the time-complexity to reach the dominance solution via the algorithm given by iterated deletion of dominated actions.

The original structure of the 11-20 game (Arad and Rubinstein 2012) is a simple game where two players submit a number from 11 to 20 and will get that number as their payoff. If their number is smaller than their opponent's by 1, they get a bonus of 20 points. This leads to a unique symmetric Nash equilibrium in (non-degenerate) mixed strategies in a similar fashion to the mixed diagonal games. Alaoui and Penta (2016) provide a modification that renders the game dominance-solvable by giving a bonus of 10 points when players choose the same number.

Diagonal games provide a tractable way to generalize the main features of the 11-20 games to an arbitrary finite number of actions and steps of deletion of dominated actions. This allows one to rank games in terms of intuitive complexity notions and make use of this variation to derive testable implications for theory and uncover novel empirical

[11] While two-player beauty contests exhibit the unappealing feature that players have a strictly dominant action, this can be fixed by requiring that players guess a fraction p of the opponent's guess, a route that is pursue by Costa-Gomes and Crawford's (2006) guessing games.

gameplay patterns. Furthermore, (symmetric) mixed diagonal games also provide a natural way to obtain a game with a unique symmetric Nash equilibrium in mixed strategies from a dominance-solvable game while retaining the predictions on level-k behavior that characterize diagonal games. Finally, the level-1 action, when defined as being the best-response to a uniform distribution, is identified in a manner that is robust to risk attitudes in diagonal games, which is not the case in the dominance-solvable variant of the 11-20 game.

A third related class of games is that of ring games by Kneeland (2015). Ring games are a defined as M players, where player i's payoffs depend only on player i's action and that of player $i+1$ (mod M), for $i = 0, 1, \ldots, M-1$. By setting payoffs such that one of the players has a dominant action, ring games enable a clean identification of cognitive levels given specific identification assumptions by having subjects play in each role.[12]

Compared to diagonal games, ring games have two major drawbacks. First, it has higher data requirements and precludes experimental implementations that are truly one-shot games, with subjects playing only once. For instance, in order to experimentally identify cognitive levels up to level 5, one needs 5 subjects playing at least once in each role, resulting in 25 data points needed per subject which are potentially affected by learning effects. Second, it provides no guidance for theorists or experimenters to choose the payoffs. Thus, while it constitutes a remarkable structure to identify level-k models, it does not yield testable implications for other models.

4.2. Experimental Implementation

Diagonal games were conceived to be a tool for theory and experiments alike. Below, I briefly discuss some factors that were explicitly taken into account and that I believe make this a good benchmark for experiments.

Comparability and Scalability. Diagonal games respond to the need of having a disciplined way to generate finite two-player dominance-solvable games that is easily scalable in the number of actions and steps of iterated deletion of dominated actions. Furthermore, taking the primitives of a given diagonal game one can easily generate another diagonal game that is more complex than the original one in sense which is not only well-defined

[12] A discussion on Kneeland's (2015) identification strategy can be found in Lim and Xiong (2016).

but also delivers interesting comparative statics across a variety of models. Having such a benchmark improves comparability across experiments and games.

Easy to Generate and Manipulate. Payoff matrices are easy to generate as the constraints on the functions v_1 and v_2 specified by the assumptions listed are straightforward to satisfy. For instance, the linear specification of v_i in (3) implied by Assumptions 1-3 is straightforward to implement and improves comparability across experiments while still allowing for some limited payoff manipulation through β_i, δ_i and $\underline{v}_i$. One can also easily break away from linearity by applying any strictly increasing function to payoffs obtained through (3). This is because Assumptions 1 and 2 (and their '-I' and '-M' variants) are *ordinal* in that if (v_1, v_2) comply with the said assumptions, so do $(f \circ v_1, g \circ v_2)$ where f and g are strictly increasing real-valued functions. Finally, going from a finite set of actions to one given by an interval (and vice-versa) is similarly straightforward.

Step Reasoning Structure. Level-k actions are identified in any of the variants of diagonal games that were introduced. Furthermore, as in the 11-20 game, it is easy to obtain a diagonal game with a unique symmetric mixed-strategy Nash equilibrium from a dominance-solvable game, but with fewer changes in the bimatrix of payoffs.

Salience and Optimality of the Dominance Solution. Dominance-solvable diagonal games avoid salience of the dominance solution as the payoffs associated to it are repeated N times in the bimatrix, precluding subjects coordinating on it for being particularly salient. This and the fact that the dominance solution payoffs lie on the Pareto frontier but are not strictly Pareto-dominant can also help minimize concerns with other-regarding preferences that are unaccounted for. It is also possible to randomize rows and columns to avoid the diagonal structure of the payoff matrix to naturally induce step-reasoning.

Description in Natural Language. Lastly, in its linear specification, one can describe the game in natural language instead of presenting the game in matrix form. Take the example of a symmetric diagonal game $\langle v, v, N \rangle$ where $v(n) = \mathbf{1}_{n \leq N+h} \cdot (\beta + n) + \mathbf{1}_{n > N+h} \cdot \underline{v}$. One succinct way to describe the game is as follows:

Both you and your opponent pick a number between 1 *and* N *and face the same rules. If you pick the largest number, you get a payoff of* $\beta + N$ *but you get a penalty equal to the difference between the number that you chose and that of your opponent. If you pick the smallest number and the difference between your opponent's number and yours is smaller than* h*, you get a payoff of* N *plus a bonus equal to that difference. If you pick the smallest number and the difference between the numbers exceeds* h*, you get a payoff of* $\underline{v}$.

5. Conclusion

The class of games introduced in this paper addresses the need to have a disciplined manner to vary the strategic environment's complexity. By focusing on dominance-solvable games, one can consider the time-complexity implied by iterated deletion of dominated strategies as a baseline for an intuitive ranking of games. Diagonal games have advantages over other existing class of games used in the literature as they enable the manipulation of a game's complexity while retaining the essential strategic features of the games.

As I illustrated, diagonal games are particularly well-suited for exploring theoretical implications related to game complexity. In particular, the proposed complexity ranking of diagonal games leads to meaningful comparative statics for a number of models of strategic interaction, with these models predicting that the players would take actions that are "farther away" from the dominance solution in more complex games. Similarly, players that face lower cognitive limitations or costs, as described by these models, would choose actions "closer" to the dominance solution.

Finally, I argued that diagonal games constitute a convenient benchmark for experiments, enabling experimenters to document new patterns within a comparable class of games. Diagonal games allow one to readily obtain dominance-solvable games that are scalable to any arbitrary finite number (or measure) of actions and steps of iterated deletion while attenuating issues such as salience of payoffs associated with specific outcomes.

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Appendix A. Omitted Proofs

Proof of Propositions 1 and 2

First, note that $\forall a_i, a'_i \in \{\max\{1, N-h\}, \ldots, N\}$, $u_i(a_i, a_j) = v_i(N - a_i + a_j) > v_i(N - a'_i + a_j) = u_i(a'_i, a_j)$ for any $a_j \in A$ whenever $a_i < a'_i$. This follows from the fact that, by Assumption 1, v_i is strictly increasing on $\{n \in \mathbb{Z} \,|\, n \leq N + h\}$ and that $N - a_i + a_j \leq a_j + h \leq N + h$ for any $a_i \geq N - h$. Therefore, a_i strictly dominates a'_i. If $N - h \leq 1$, this implies that action 1 is strictly dominant and claim (i) of Proposition 1 follows and that $a_i \rhd_{i,A} a'_i$ for any $a_i, a'_i \in A$ such that $a_i \leq a'_i$. To show claim (ii) of the same proposition, let us consider the case where $N - h > 1$. Then, $u_i(N-h, N) = v_i(N+h) > u_i(a_i, N)$ for any $a_i \neq N - h$ and, consequently, $a_i \rhd_{i,A} a'_i$ and $A_i^1 = \{1, \ldots, \max\{1, N-h\}\}$. As the above holds for both $i = 1,2$, consider the restricted game after deleting the strictly dominated actions $\{N-h+1, \ldots, N\}$. Note that the resulting game is still a diagonal game characterized by $\langle v'_1, v'_2, N' \rangle$, where $N' = N - h$ and $v'_i(n) = v_i(n+h)$, for $n \in \mathbb{Z}$. Furthermore, $h' = \arg\max_n v'_i(n) - N' = \arg\max_n v_i(n+h) - N' = N - N' = h$. Then, similarly to above, we have that $\forall a_i, a'_i \in \{\max\{1, N'-h\}, \ldots, N'\}$, $u_i(a_i, a_j) = v'_i(N' - a_i + a_j) > v'_i(N' - a'_i + a_j) = u_i(a'_i, a_j)$ for any $a_j \in A$ whenever $a_i < a'_i$. Hence, if $a_i < a'_i$, a_i strictly dominates a'_i in this restricted game and $a_i \rhd_{i,A} a'_i$. By noting that $N' = N - h$, we then get that $\forall a_i, a'_i \in \{\max\{1, N - 2\cdot h\}, \ldots, N-h\}$ such that $a_i \leq a'_i$, $a_i \rhd_{i,A} a'_i$. Then, claim (ii) of Proposition 1 and Proposition 2 follow by an induction argument. As at each step of deletion, h actions are eliminated, there will be exactly $T = \lceil (N-1)/h \rceil$ steps.

To see that diagonal games complying with Assumption 1 are supermodular note that $(A, \geq)$ is a chain. Then, for $i = 1,2$, u_i is trivially continuous and quasisupermodular in a_i. I now will show that it has the single-crossing property. That is, for $a'_i > a_i$ and $a'_j > a_j$, if $u_i(a'_i, a_j) - u_i(a_i, a_j) \geq 0$, then $u_i(a'_i, a'_j) - u_i(a_i, a'_j) \geq 0$. First, note that $u_i(a'_i, a_j) - u_i(a_i, a_j) = v_i(N - a'_i + a_j) - v_i(N - a_i + a_j) \geq 0 \implies N - a_i + a_j > N + h$ as if otherwise, by Assumption 1, $v_i(N - a'_i + a_j) - v_i(N - a_i + a_j) < 0$ as v_i is strictly increasing up to $N + h$. Then, let us consider two cases: $N - a'_i + a'_j \leq N + h$ and $N - a'_i + a'_j > N + h$. If $N - a'_i + a'_j \leq N + h$, then $N - a_i + a'_j > N - a_i + a_j > N + h \geq N - a'_i + a'_j > N - a'_i + a_j$. Hence, we have $v_i(N - a_i + a'_j) \leq v_i(N - a_i + a_j) \leq v_i(N - a'_i + a_j) < v_i(N - a'_i + a'_j)$, where the first inequality follows from the fact that v_i is single-peaked, with the unique local maximum at $N + h$, the second from the fact that, by assumption, $v_i(N - a'_i + a_j) \geq v_i(N - a_i + a_j)$, and the third as v_i is strictly increasing up to the $N + h$ and $N - a'_i + a_j < N - a'_i + a'_j \leq N + h$. Thus, if $u_i(a'_i, a_j) - u_i(a_i, a_j) \geq 0$ and $N - a'_i + a'_j \leq N + h$, then $u_i(a'_i, a'_j) - u_i(a_i, a'_j) \geq 0$. Now consider the case where $N - a'_i + a'_j > N + h$. Then, as $N - a_i + a'_j > N - a'_i + a'_j > N + h$, it immediately follows from v_i being single peaked, with its unique local maximum at $N + h$, that $u_i(a'_i, a'_j) = v_i(N - a'_i + a'_j) \geq v_i(N - a_i + a'_j) = u_i(a_i, a'_j)$.

Proof of Propositions 3 and 4

The level-1 action is given by the best-response to an opponent who uniformly randomizes over A. The sum of the payoffs to action $N - h$ are $\sum_{m=1}^{N} v_i(m + h)$. For any $n > N - h$, by , 2 we know that $N - h$ strictly dominates n. For any $n < N - h$, let $\ell > 0$ be such that $n = N - h - \ell$ and note that the sum of payoffs to action n is given by $\sum_{m=1}^{N} v_i(N - n + m) = \sum_{m=1+\ell}^{N+\ell} v_i(h + m)$. Then, $\sum_{m=1}^{N} v_i(m + h) - \sum_{m=1}^{N} v_i(N - n + m) = \sum_{m=1}^{\ell} v_i(h + m) - \sum_{m=1+N}^{N+\ell} v_i(h + m) = \sum_{m=1}^{\ell}(v_i(h+m) - v_i(N+h+m)) = \sum_{m=1}^{\ell}(v_i(h+m) - \arg\min_{n' \geq 1} v_i(n')) > 0$. Moreover, as the same holds for any strictly increasing transformation of v_i, the level-1 action is identified in a manner that does not depend on player i's risk attitudes, where v_i to denote monetary payoffs instead of utility. Then, it is straightforward to check that, for any $k = 1, \ldots, \lceil (N - 1)/h \rceil - 2$, the unique best-response to $N - k \cdot h$ is $N - (k + 1) \cdot h$ as $u_i(N - (k + 1) \cdot h, N - k \cdot h) = v_i(N + h)$. Furthermore, the unique best-response to $N - (\lceil (N - 1)/h \rceil - 1) \cdot h$ is 1, given that $N + N - (\lceil (N - 1)/h \rceil - 1) \cdot h \leq N + h + 1$ and thus $\forall a_i \in A$, $N - (\lceil (N - 1)/h \rceil - 1) \cdot h - a_i \leq N + h$, implying that $u_i(a_i, N - (\lceil (N - 1)/h \rceil - 1) \cdot h)$ is strictly decreasing in a_i by Assumption 1. This concludes the proof of Proposition 3. Proposition 4 follows immediately from the fact that, after one step of deletion of dominated actions, the restricted game is still a diagonal game characterized by $\langle v'_1, v'_2, N' \rangle$, where $N' = N - h$, $v'_i(n) = v_i(n + h)$, for $n \in \mathbb{Z}$, and $h' = \arg\max_n v'_i(n) - N' = \arg\max_n v_i(n + h) - N' = N - N' = h$. Thus, the level-1 this restricted game corresponds to $N - 2 \cdot h$, the level-2 action in the original game. Following an argument by induction, the claim in Proposition 4 ensues.

Proof of Lemma 1

The result is implied by a special case of Lemma 3, which is proved in Appendix A.1 where I discuss the implications of asymmetric anchors within the context of the endogenous depth of reasoning model.

Proof of Lemma 2

Under the maximum gain specification in (2), the value of reasoning in diagonal games satisfying Assumptions 1 and 2 is exactly

$$V_i(k) = \begin{cases} \max_{a_i,a_j} v_i(N - a_i + a_j) - v_i(N - a_0^i + a_j), & \text{if } a_0^i \geq N-h \text{ and } k=1 \\ v_i(N+h) - v_i(N+h+1), & \text{if otherwise} \end{cases}$$

To see this, note that when $a_k^i < N-h$, then $\max_{a_i,a_j} u_i(a_i,a_j) - u_i(a_k^i,a_j) = v_i(N - a_i + a_j) - v_i(N - a_k^i + a_j)$. Now, given the assumptions on v_i, we have that $v_i(N+h) \geq v_i(n) \geq v_i(N+h+1)$, $\forall n = 1,\ldots,2N$. Then, let $a_j = a_k^i + h + 1 \in \{1,\ldots,N\}$ and $a_i = a_j - h \in \{1,\ldots,N\}$. Consequently, $v_i(N - a_i + a_j) - v_i(N - a_k^i + a_j) = v_i(N+h) - v_i(N+h+1)$. Thus, if $a_0^i < N-h$, $V_i(1) = v_i(N+h) - v_i(N+h+1)$, with $v_i(N+h+1) = \underline{v}_i$, by Assumption 2. As, from Lemma 1, for $k \geq 2$, $a_{i,k}^i < N-h$, regardless of a_0^i, we also have that $V_i(k) = v_i(N+h) - v_i(N+h+1)$ for any $k \geq 2$. Finally, the claim follows from the fact that $V_i(1)\max_{a_i,a_j} u_i(a_i,a_j) - u_i(a_k^i,a_j) \leq \max_{a_i,a_j} u_i(a_i,a_j) - \min_{a_i,a_j} u_i(a_i,a_j) = v_i(N+h) - v_i(N+h+1)$.

Proof of Propositions 5 and 6

Proposition 5 follows directly from Lemmata 2 and 2. Proposition 6 follows from the fact that if $\hat{k}_i \geq 1$ and $c_i \leq \tilde{c}_i$, the $\hat{k}_i \geq \tilde{k}_i$ as, by Lemma 2, V_i is weakly decreasing. Moreover, as $\tilde{\Gamma} \triangleright_C \Gamma$ and $a_0^i \triangleright_A \tilde{a}_i^0$, it then follows that $a_i^* \triangleright_A \tilde{a}_i^*$.

Proof of Proposition 7

Recall that, for any Dirichlet prior with parameters $\alpha \in \mathbb{R}_{++}^N$, upon observing $a_j \sim \sigma_j$, player i's posterior will still be a Dirichlet distribution, now with parameters $\alpha + (\mathbf{1}_{a_j=n})_{n=1}^N$. Moreover, by linearity, we have that

$$\int_{\sigma'_j \in \Delta(A)} u_i(\sigma'_i, \sigma'_j) d(\mu_i \mid a_j^k)(\sigma'_j) = u_i(\sigma'_i, \int_{\sigma'_j \in \Delta(A)} \sigma'_j d(\mu_i \mid a_j^k)(\sigma'_j))$$
$$= u_i(\sigma'_i, \mathbb{E}_{\mu_i}[\sigma'_j | a_j^k]),$$

which implies that, when choosing, player i only cares about the posterior mean. Using Dirichlet priors also simplifies the matter, as for any Dirichlet prior with parameters $\alpha \in \mathbb{R}^N_{++}$, the posterior mean is just $\alpha / \|\alpha\|_1$, where $\|\cdot\|_1$ denotes the 1-norm. In our case, denote by $s_n(a_j^k)$ the number of n-valued observations in a sample a_j^k,[13] this will mean that for any sample a_j^k, player i's best-response is just $\arg\max_{\sigma'_i \in \Delta(A)} \sum_{n=1}^N \frac{1+s_n(a_j^k)}{N+k} u_i(\sigma'_i, n)$.

Note that $\forall n + m \leq N - h$,

$$\begin{aligned}
& \mathbb{E}_{\mu_i}[u_i(n, \sigma_j) \mid x_i^k] \geq \mathbb{E}_{\mu_i}[u_i(n+m, \sigma_j) \mid x_i^k] \\
\iff & \sum_{\ell=1}^{n+h} \sigma_j(\ell) \cdot v_i(N+\ell-n) + \sum_{\ell=n+1+h}^{n+m+h} \sigma_j(\ell) \cdot \underline{v}_i + \sum_{\ell=n+m+h+1}^{N} \sigma_j(\ell) \cdot \underline{v}_i \geq \\
\geq & \quad \sum_{\ell=1}^{n+h} \sigma_j(\ell) \cdot v_i(N+\ell-(n+m)) + \sum_{\ell=n+1+h}^{n+m+h} \sigma_j(\ell) \cdot v_i(N+\ell-(n+m)) + \sum_{\ell=n+m+h+1}^{N} \sigma_j(\ell) \cdot \underline{v}_i \\
\iff & \sum_{\ell=1}^{n+h} \sigma_j(\ell) \cdot [\beta_i + (N+\ell-n) \cdot \delta_i] + \sum_{\ell=n+h+1}^{n+m+h} \sigma_j(\ell) \cdot \underline{v}_i \\
\iff & \sum_{\ell=1}^{n+h} \sigma_j(\ell) \cdot [\beta_i + (N+\ell-(n+m)) \cdot \delta_i] + \sum_{\ell=n+h+1}^{n+m+h} \sigma_j(\ell) \cdot \underline{[}\beta_i + (N+\ell-(n+m)) \cdot \delta_i] \\
\iff & \sum_{\ell=1}^{n+h} \sigma_j(\ell) \cdot m \geq \sum_{\ell=n+h+1}^{n+m+h} \sigma_j(\ell) \cdot \underline{[}\frac{\beta_i + \delta_i - \underline{v}_i}{\delta_i} + (N+\ell-(n+m)-1)] \\
\iff & m \cdot (n+h) + m \cdot \sum_{r=1}^{k} \sum_{\ell=1}^{n+h} \mathbf{1}_{x_{i,r}=\ell} \geq \\
\geq & \quad m \cdot (N+h) - \frac{m \cdot (m+1)}{2} + m \frac{\beta_i + \delta_i - \underline{v}_i}{\delta_i} + \sum_{\ell=n+h+1}^{n+m+h} \sum_{r=1}^{k} \mathbf{1}_{x_{i,r}=\ell} \cdot \left[\frac{\beta_i + \delta - \underline{v}_i}{\delta_i} + (N+\ell-(n+m)-1) \right] \\
\iff & m \cdot (n-N) + m \cdot \sum_{r=1}^{k} \sum_{\ell=1}^{n+h} \mathbf{1}_{x_{i,r}=\ell} \geq \\
\geq & \quad -\frac{m \cdot (m+1)}{2} + m \frac{\beta_i + \delta_i - \underline{v}_i}{\delta_i} + \sum_{\ell=n+h+1}^{n+m+h} \sum_{r=1}^{k} \mathbf{1}_{x_{i,r}=\ell} \cdot \left[\frac{\beta_i + \delta - \underline{v}_i}{\delta_i} + (N+\ell-(n+m)-1) \right]
\end{aligned}$$

[13] That is, $s_n(a_j^k) = \sum_{\ell=1}^k \mathbb{1}_{a_{j,\ell}=n}$.

Let $m = 1$. Note that

$$n-(N-1)+k \geq (n-(N-1))+\cdot\sum_{r=1}^{k}\sum_{\ell=1}^{n+h}\mathbf{1}_{x_{i,r}=\ell}$$
$$\frac{\beta_i+\delta_i-\underline{v}_i}{\delta_i}+\sum_{r=1}^{k}\mathbf{1}_{x_{i,r}=n+h+1}\cdot\left[\frac{\beta_i+\delta-\underline{v}_i}{\delta_i}+(N+\ell-(n+1)-1)\right]$$
$$\geq \frac{\beta_i+\delta_i-\underline{v}_i}{\delta_i}$$

Hence, $\mathbb{E}_{\mu_i}[u_i(n,\sigma_j)\mid x_i^k] \geq \mathbb{E}_{\mu_i}[u_i(n+1,\sigma_j)\mid x_i^k]$ only if $n \geq N-1-k$. As no action $a_i \in \{N-h+1,\dots,N\}$ will be chosen given that they are strictly dominated by $N-h$, we have that at any k-sampling $\sigma_i(a_i)=0$ for $a_i > N-h$ and $a_i < N-1-\max\{k,h\}$.

Let $n = N-1-k+M$ and suppose that $N-1-k+M < N-h$. Then, $\forall m \geq 1$,

$$m\cdot(n-N)+m\cdot\sum_{r=1}^{k}\sum_{\ell=1}^{n+h}\mathbf{1}_{x_{i,r}=\ell} \geq$$
$$\geq \quad -\frac{m\cdot(m+1)}{2}+m\frac{\beta_i+\delta_i-\underline{v}_i}{\delta_i}+\sum_{\ell=n+h+1}^{n+m+h}\sum_{r=1}^{k}\mathbf{1}_{x_{i,r}=\ell}\cdot\left[\frac{\beta_i+\delta-\underline{v}_i}{\delta_i}+(N+\ell-(n+m)-1)\right]$$
$$\iff \quad n-N+k-M \geq -\frac{m+1}{2}$$
$$\implies \quad n \geq N+M-k-1.$$

Consequently, if $\sigma_j(n)=1$, $\mathbb{E}_{\mu_i}[u_i(n,\sigma_j)\mid x_i^k] \geq \mathbb{E}_{\mu_i}[u_i(n+m,\sigma_j)\mid x_i^k]$, $\forall m = 1,2,\dots,N-n$. As such, there is a k-sampling equilibrium with $\sigma_i(n)=1$, for $i=1,2$, and, from the above, it is the $\rhd_{A,FOSD}$-largest k-sampling equilibrium. If instead $N-1-k+M > N-h$, then the unique k-sampling equilibrium is $\sigma_i(N-h)=1$, for $i=1,2$. This concludes the proof to claim (i).

Claim (ii) follows from the fact that $\forall n \leq N-h-1$, $\mathbb{E}_{\mu_i}[u_i(n,\sigma_j)\mid x_i^k] \geq \mathbb{E}_{\mu_i}[u_i(n+1,\sigma_j)\mid x_i^k] \implies n-(N-1)+k \geq \frac{\beta_i+\delta_i-\underline{v}_i}{\delta_i}$. As $n+1 \leq N-h$ and $n-(N-1)+k \geq \frac{\beta_i+\delta_i-\underline{v}_i}{\delta_i}$ imply that $n+1+k-\frac{\beta_i+\delta_i-\underline{v}_i}{\delta_i}-h \geq N-h \geq n+1$, then as $k-\frac{\beta_i+\delta_i-\underline{v}_i}{\delta_i}-h \leq 0$ by assumption, we immediately have that $\sigma_i(N-h-1)+\sigma_i(N-h)=1$ and that, if $h>k$ of $\beta_i+\delta_i > \underline{v}_i$, then $\sigma_i(N-h)=1$, concluding the proof.

Proof of Proposition 8

The proof for (i) follows from the fact that if $v_i(1) < v_i(N+h')$, then $N-h'$ is still the unique best-response to a uniform distribution by a similar argument as that in the proof of Proposition 3 plus the fact that $u_i(N-h',m) \geq u_i(N,m+1)$ for any $m = 1,...,N-1$ and $u_i(N-h',N) = v_i(N+h') > v_i(1) = u_i(N,1)$. If instead $v_i(1) \geq v_i(N+h')$, then a level-1 player chooses either N or $N-h$. As the unique best-response to N is $N-h$, then (i) follows as level $k \geq 1$ are restricted to choosing actions, that is, degenerate strategies.

I will now prove (ii). As by assumption the game is symmetric, I will drop the indices denoting players.
I first prove (ii) for the case where $h = 1$ and then show how the case where $h > 1$ can be collapsed to one similar to $h = 1$.

When $h = 1$, it should be easy to see that all actions are chosen by players of some level $k \in \mathbb{N}$. First we show that any symmetric Nash equilibrium $(\sigma_1, \sigma_2) = (\sigma, \sigma)$ must then have full support. Suppose that $\sigma(N) = 0$. Then, $u(N-1,\sigma) < u(N-2,\sigma)$ as $u(n,m) > u(n+1,m)$ $\forall n = 1,..,N-1$ and $m \leq n+1$. This then implies also that $\sigma(N-1) = 0$. By induction it would be the case that $\sigma(1) = 1$, but then there would be a profitable deviation to $\sigma(N) = 1$. Then note that $\sigma(1) > 0$ as if otherwise $u(N,\sigma) < u(N-1,\sigma)$ which would contradict that any symmetric equilibrium places positive probability on action N. Similarly, if $\sigma(2) = 0$, then $u(1,\sigma) < u(2,\sigma)$ and it must be that $\sigma(1) = 0$, which was just shown could not be. By induction, the argument generalizes to $n = 3,...,N-1$. Hence, any symmetric equilibrium must have full support.

Now suppose for the purpose of contradiction that there are two symmetric Nash equilibria σ and σ', $\sigma \neq \sigma'$. From the above, it must be that $\sigma, \sigma' \in \text{int}(\Sigma)$. Let $\overline{\sigma}$ denote the element of the singleton $\{\sigma - \lambda(\sigma - \sigma'), \lambda \in \mathbb{R}_+\} \cap \partial\Sigma$, where $\partial\Sigma$ denotes the boundary of the $(N-1)$-dimensional simplex, and, similarly, let $\underline{\sigma}$ denote the element of the singleton $\{\sigma + \lambda(\sigma - \sigma'), \lambda \in \mathbb{R}_+\} \cap \partial\Sigma$.
I will now show that there are $\underline{n}$ and $\overline{n}$ such that $\overline{\sigma} \in \Sigma^*(\overline{n}) \setminus \Sigma^*(\underline{n})$ and $\underline{\sigma} \in \Sigma^*(\underline{n}) \setminus \Sigma^*(\overline{n})$, where $\Sigma^*(n) := \{\hat{\sigma} \in \Sigma : u(n,\hat{\sigma}) = \max_{\sigma'' \in \Sigma} u(\sigma'', \hat{\sigma})\}$, the set of distributions $\hat{\sigma}$ such that n is a best-response against $\hat{\sigma}$.

First, note that as $\overline{\sigma},\underline{\sigma} \in \partial\Sigma$, then $\overline{\sigma},\underline{\sigma} \notin \bigcap_{n=1}^{N} \Sigma^*(n)$ as $\forall \sigma'' \in \Sigma$, $\forall n$ such that $\sigma''(n-1 \mod_1 N) > \sigma''(n \mod_1 N) = 0$, it will be the case that $u(n,\sigma) < u(N,\sigma)$ when $n = 1$ and $u(n,\sigma) < u(n-1,\sigma)$, when otherwise, which follows immediately by the structure of payoffs.

Let $\overline{N} := \{n : \overline{\sigma} \in \Sigma^*(n)\}$ and $\underline{N} := \{n : \underline{\sigma} \in \Sigma^*(n)\}$. I will show that $\overline{N} \setminus \underline{N} \neq \emptyset$ and that $\underline{N} \setminus \overline{N} \neq \emptyset$.

Suppose that $\overline{N} = \underline{N}$. Then $\forall n \in \overline{N}$, $n' \notin \overline{N}$, $\forall \lambda \in (0,1)$, $u(n, \lambda\overline{\sigma}+(1-\lambda)\underline{\sigma}) > u(n', \lambda\overline{\sigma}+(1-\lambda)\underline{\sigma})$. As σ is a convex combination of $\overline{\sigma}$ and $\underline{\sigma}$, by linearity we have that $u(n,\sigma) > u(n',\sigma)$, which contradicts the fact that σ, as a symmetric Nash equilibrium, has full support. Suppose instead that $\overline{N} \subset \underline{N}$. Then, $\forall n \in \overline{N}$ and $n' \in \underline{N} \setminus \overline{N}$, $\forall \lambda \in (0,1)$, again, $u(n, \lambda\overline{\sigma} + (1-\lambda)\underline{\sigma}) > u(n', \lambda\overline{\sigma} + (1-\lambda)\underline{\sigma})$, implying $u(n,\sigma) > u(n',\sigma)$, a contradiction. The argument is the same for $\overline{N} \supset \underline{N}$. Consequently, $\overline{N} \setminus \underline{N} \neq \emptyset$ and that $\underline{N} \setminus \overline{N} \neq \emptyset$.

Now let $\overline{n} \in \overline{N} \setminus \underline{N}$ and that $\underline{n} \in \underline{N} \setminus \overline{N}$ and let $\Sigma_{\overline{n},\underline{n}}$ denote the $(N-2)$-dimensional hyperplane of distributions $\hat{\sigma}$ such that $u(\overline{n},\hat{\sigma}) = u(\underline{n},\hat{\sigma})$. Then, $\overline{\sigma}$ and $\underline{\sigma}$ lie on the two different half-spaces of the $(N-1)$-dimensional simplex separated by the hyperplane $\Sigma_{\overline{n},\underline{n}}$. As such, there is a unique $\gamma \in (0,1)$ such that $\gamma\overline{\sigma} + (1-\gamma)\underline{\sigma} \in \Sigma_{\overline{n},\underline{n}}$. Both σ and σ', the symmetric Nash equilibria, are linear combinations of $\overline{\sigma}$ and $\underline{\sigma}$ and $\sigma,\sigma' \in \Sigma_{\overline{n},\underline{n}}$. Full support implies indifference between any two strategies. Then $\sigma \neq \sigma'$ contradicts uniqueness of γ. Hence, when $h = 1$, there is a unique symmetric Nash equilibrium and it has support on the set of actions chosen by level-k players.

I now extend this result to $h > 1$. Note that the actions $a_i = N-h+1,\dots,N-1$ are strictly dominated by action $N-h$. Once one eliminates such actions, we obtain $A^1 = \{1,2,\dots,N-h,N\}$. Then, $\forall a_i = \{N-2h+1,\dots,N-1-h\}$ and $\forall a_j \leq N-h$, $u(a_i,a_j) = v(N-(N-2h+\ell_i)+(N-h-\ell_j)) = v_i(N+h-\ell_i-\ell_j) < v(N+h-\ell_j) = u(N-2h,a_j)$, where I wrote $a_i = N-2h+\ell_i$ and $a_j = N-h-\ell_j$. Moreover, $u(a_i,N) = u(N-2h,N)$. As $\sigma(N) < 1$ in any equilibrium, it must be that $\sigma(a_i) = 0$ as $u(a_i,\sigma) < u(N-2h,\sigma)$. Then let $A^2 = \{1,2,\dots,N-2h,N-h,N\}$.

Similarly, $\forall a_i \in \{N-3h+1,\dots,N-1-2h\}$ and $\forall a_j \leq N-2h$, $u(a_i,a_j) < u(N-3h,a_j)$ and $u(a_i,N) = u(a_i,N-h) = u(N-3h,N) = u(N-3h,N-h)$. Suppose that σ has support on $\{N-h,N\}$. Then $u(N-h,\sigma) > u(N,\sigma)$, which contradicts players being indifferent between N and $N-h$. Then, $\sigma(a_i) = 0$ and let $A^3 = \{1,2,\dots,N-3h,N-2h,N-h,N\}$.

Iterating the argument will take us to considering the restricted game $A^k = \{1,2,\dots,N - k\cdot h, N-(k-1)\cdot h, N-(k-3)\cdot h,\dots,N-h,N\}$, where $N - k\cdot h \leq 1+h$. Let us argue that $\sigma(a_i) = 0$ for $a_i = 2,3,\dots,N-k\cdot h-1$. First, note that $\forall a_j \leq 1+h$, $u(1,a_j) > u(a_i,a_j)$ and for all the remaining $a_j \in A^k$ we have that $u(1,a_j) = u(a_i,a_j)$. Consider the case where it is an equilibrium to have $\sum_{\ell=0}^{k}\sigma(N-\ell\cdot h) = 1$. Then $(u(N,\sigma) < u(N-h,\sigma) \implies \sigma(N) = 0) \implies (u(N-h,\sigma) < u(N-2h,\sigma) \implies \sigma(N-h) = 0) \implies \cdots \implies \sigma(N-k\cdot h) = 1$ and $u(1,\sigma) > u(N-k\cdot h,\sigma)$. Hence, for any symmetric equilibrium, it must be that it only places positive probability on $\{\max\{1, N-k\cdot h\}, k = 1,2,\dots\}$.

Then, consider the game restricted to $\tilde{A} = \{\max\{1,N-k\cdot h\}, k=1,2,\dots\}$. It is straightforward to check that this is a diagonal game with parameters $\tilde{N} = |\{\max\{1,N-k\cdot h\},k=1,2,\dots\}|$ and $\tilde{h} = 1$. Hence, there is a unique symmetric Nash equilibrium in this restricted game and it has full support.

Finally, note that $\{\max\{1,N-k\cdot h\},k=1,2,\dots\}$ coincides with the set of actions player by level-k players, $k \geq 1$.

A.1. On Asymmetric Anchors in the Endogenous Depth of Reasoning Model

The assumption that $a^i_{i,0} = a^i_{j,0}$ is indeed necessary for the results in Section 2.2. Once we step away from this assumption, one may get that reasoning more leads to "worse" actions, i.e. the players choosing larger numbers. This observation follows from this next lemma.

Lemma 3. Let Γ be a diagonal game and suppose Assumption 1 holds. Then, for any $k \in \mathbb{N}_0$, $a_{i,1+2\cdot k} = \max\{1, a^i_{j,0} - (1+2\cdot k)h\}$ and $a_{i,2+2\cdot k} = \max\{1, a^i_{j,0} - (2+2\cdot k)h + (a^i_{i,0} - a^i_{j,0})\}$.

Proof. Without loss of generality, let $a^i_{i,0} = a^i_{j,0} + m$. Note that $a^i_{i,1} = \max\{1, a^i_{j,0} - h\}$ and that $a^i_{i,2} = \max\{1, a^i_{j,1} - h\} = \max\{1, a^i_{i,0} - 2h\} = \max\{1, a^i_{j,0} - 2h + m\}$. This is a consequence of the fact that for any action of the opponent a_j, the unique best-response is given by $\max\{1, a_j - h\}$. Then, the claim follows immediately by induction. □

Lemma 3 implies that if N is sufficiently large relative to h, $a^i_{i,0} - 2h > 1$ and $a^i_{i,0} - a^i_{j,0} > h$, then $a^i_{i,2} > a^i_{i,1}$. Therefore, with a judicious choice of the cost of reasoning c_i, it is possible to find an increase in payoffs that results in changing $\hat{k}_i$ from 1 to 2 and inducing player

i to choose a larger number. A similar argument can be made to show that this is not specific to diagonal games. For instance, one can then show that Alaoui and Penta's (2016) assumption that $a^i_{i,0} = a^i_{j,0}$ was indeed necessary for results in Proposition 4, highlighting the importance of the choice of anchors in the endogenous depth of reasoning model.