ECON0106: **Microeconomics**

1. Choice, Preferences, Utility[*]

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1. Overview

Economic theory studies the behaviour of agents: to predict what we expect there to happen, to explain why we observe a particular regularity, to recommend a particular course of action. At its core we find a model, a stylized but informative representation of the situation being studied. Our goal is to develop building blocks that can and have been used to model a wide variety of questions, e.g., consumer demand and firm pricing, student applications to university, voting, technology adoption, hospital residency program management, etc.

The three main approaches that have been taken are to represent an agent's behaviour by means of their choices, their preferences, or a utility function. We often work directly with the assumption that agents choices are described by utility maximisation: agents choose an alternative x from a set of feasible alternatives S to maximise their utility u.

However, utility functions are not directly observable: we just observe their choices. How then can we make sure that the utility function we are using is the right one? By assuming a particular utility function, we are implicitly making assumptions on how agents behave, on their preferences, that may or may not be reasonable assumptions, depending on the application at hand. Hence, we will be paying some attention on how properties of utility functions relate to properties of the agents preferences, and how those, in turn, relate to properties of their choices. This will enable us to *test* our models, as we can *identify* the assumptions underlying them. On a more pragmatic level, while assumptions are often for the sake of tractability — a model is, after all, a simplified description of reality — studying their properties and their *empirical content* allows us to better understand the limitations of our models.

2. Choice and Preferences

We will start by fixing a finite set of alternatives X and consider all possible subsets $2^X := \{A \mid A \subseteq X\}$. The agent can then choose from a subset of alternatives, $A \in 2^X$, which we model via

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a choice function:

Definition 1. A **choice function** is a function $C : 2^X \to 2^X$ such that $C(A) \subseteq A\ \forall A \in 2^X$. We further require choice functions to be **nonempty**, that is, $\forall A \neq \emptyset$, $C(A) \neq \emptyset$.[1]

In short, a choice function determines the agent's choices in every possible situation.

Another way to model behaviour is by considering preference relations on X. We say that $\succsim \subseteq X \times X$ is a **binary relation** on X and if $(x, y) \in \succsim$, we will often write $x \succsim y$ (or $y \precsim x$). Let us introduce some properties that binary relations can satisfy.

Definition 2. We say that a binary relation $\succsim$ on X is

- **reflexive** iff, $\forall x \in X$, $x \succsim x$;
- **transitive** iff, $\forall x, y, z \in X$, $x \succsim y$ and $y \succsim z$ implies $x \succsim z$;
- **negatively transitive** iff, $\forall x, y, z \in X$, $x \succsim y$, then $x \succsim z$ or $z \succsim y$;
- **complete**[2] iff, $\forall x, y \in X$, $x \succsim y$ or $y \succsim x$;
- **antisymmetric** iff, $\forall x, y \in X$, $x \succsim y$ and $y \succsim x$ implies $x = y$;
- **symmetric** iff, $\forall x, y \in X$, $x \succsim y$ implies $y \succsim x$;
- **asymmetric** iff, $\forall x, y \in X$, $x \succsim y$ implies $\neg(y \succsim x)$.

The binary relation is then given different names when it satisfies different properties:

Definition 3. A binary relation $\succsim$ is called

(i) a **preorder** iff it is reflexive and transitive;

(ii) a **partial order** iff it is reflexive, transitive, and antisymmetric (an antisymmetric preorder);

(iii) a **linear order** (or total order) iff it reflexive, transitive, antisymmetric, and complete (a complete partial order).

In each of those cases, $(X, \succsim)$ is called (i) a preordered set, (ii) a partially ordered set, and (iii) and linearly or totally ordered set. Some examples for $(X, \succsim)$: (i) people in a room and their height (having the same height does not mean they are the same person), (ii) $\mathbb{R}^n$ and the natural product order $x \geq y \iff x_i \geq y_i, i = 1, .., n$, (iii) $\mathbb{R}$ and the natural order.

Throughout, we will assume that **preference relations** are complete and transitive and when $x \succsim y$ we say that x is *weakly preferred* to y. We allow the agent to be *indifferent* between two

[1] This does not mean that we don't allow the agent to "choose/do nothing"; rather, that we will make "choose/do nothing" an element of X.

[2] In order theory, especially outside economics, you may also find this property being called (strongly) connected, total, or connex.

alternatives x and y, that is, $x \succsim y$ and $y \succsim x$, in which case we write $x \sim y$. Note that $x \sim y$ does not imply $x = y$: the agent may be indifferent between an apple and a banana, but that does not mean that they are the same element (this is why we don't require that $\succsim$ be antisymmetric). We say that x is *strictly preferred* to y if $x \succsim y$ and $\neg(y \succsim x)$, and we write $x \succ y$.[3] Often, $\succ$, the asymmetric or strict part of $\succsim$, is called a strict preference relation, whereas $\sim$ is called an indifference relation, corresponding to the symmetric part of $\succsim$. Note that the asymmetric ($\succ$) and symmetric ($\sim$) parts of $\succsim$ can be defined for any binary relation $\succsim \subseteq X \times X$, and that $\succsim = \succ \cup \sim$.

The next proposition shows that if we are given a strict preference relation, we can recover the original preferences:

Proposition 1. *A binary relation $\succsim \subseteq X \times X$ is complete and transitive only if its asymmetric part, $\succ \subseteq X \times X$, is asymmetric and negatively transitive. A binary relation $\succ \subseteq X \times X$ is asymmetric and negatively transitive only if there is $\succsim \subseteq X \times X$ such that $\succ \subseteq \succsim$, $\succ$ is the asymmetric part of $\succsim$, and $\succsim$ is complete and transitive.*

Exercise 1. *(i) If you are given a strict preference relation $\succ$, how do you recover (construct) a consistent weak preference relation $\succsim$?*

(ii) Prove Proposition 1.

3. Revealed Preference

For a preference relation $\succsim$ on X, define, for every $A \in 2^X$, $\arg\max_{\succsim} A := \{x \in A \mid x \succsim y \text{ for all } y \in A\}$, the set of maximisers in A, that is, the most preferred elements in A. We want to understand when can we represent an agent's choices as being driven by preference maximisation.

Let us note some properties of $\arg\max_{\succsim}$:

Proposition 2. *Let $\succsim \subseteq X \times X$ be a preference relation. The following properties hold:*

(i) If $B \subseteq A \subseteq X$, then for any $x \in \arg\max_{\succsim} A$ and $y \in \arg\max_{\succsim} B$, $x \succsim y$.

(ii) If $x \in B \subseteq A \subseteq X$, and $x \in \arg\max_{\succsim} A$, then $x \in \arg\max_{\succsim} B$.

(iii) For any nonempty $A \subseteq X$, $\arg\max_{\succsim} A \neq \emptyset$.

(iv) For $x, y \in A \subseteq X$, $x \sim y$ and $\{x, y\} \cap \arg\max_{\succsim} A \neq \emptyset$ if and only if $\{x, y\} \subseteq \arg\max_{\succsim} A$.

Proof. (i) As $x \in \arg\max_{\succsim} A \iff x \succsim z \; \forall z \in A$, and $y \in B \subseteq A$, the result follows.

(ii) As $x \in \arg\max_{\succsim} A \iff x \succsim z \; \forall z \in A$ and $B \subseteq A$, then $x \succsim z \; \forall z \in B \iff x \in \arg\max_{\succsim} B$.

[3] We will also equivalently use the expressions "x (weakly/strictly) dominates y".

(iii) As X is finite, A is finite. For any $A \in 2^X$ such that $|A| = 1$, then $A = \arg\max_{\succsim} A$ as $x \succsim x$ (by completeness), and therefore $x \sim x$. For the purpose of induction, suppose that for any $B \in 2^X$ such that $B \neq \emptyset$ and $|B| = n \geq 1$, $\arg\max_{\succsim} B \neq \emptyset$. Take any $A \in 2^X$ such that $|A| = n+1$; we want to show that $\arg\max_{\succsim} A \neq \emptyset$. By definition, $A = B \cup \{x\}$, where $|B| = n$, and, for any $y, z \in \arg\max_{\succsim} B \neq \emptyset$, by completeness, $y \succsim x$ or $x \succsim y$. If the former, then we have that $y \in \arg\max_{\succsim} A$, as $y \succsim z \ \forall z \in B$ and $y \succsim x$. If the latter, then as $x \succsim y$ and $y \in \arg\max_{\succsim} B \iff y \succsim z \ \forall z \in B$, by transitivity, $x \succsim z \ \forall z \in B$, and hence $x \in \arg\max_{\succsim} A$.

(iv) Let $\{x, y\} \subseteq A$, $x \sim y$ and $\{x, y\} \cap \arg\max_{\succsim} A \neq \emptyset$. Without loss of generality, suppose $x \in \arg\max_{\succsim} A$. As $y \sim x \implies y \succsim x \succsim z \ \forall z \in A$, by transitivity $y \succsim z \ \forall z \in A \iff y \in \arg\max_{\succsim} A$. For the other direction, if $\{x, y\} \subseteq \arg\max_{\succsim} A$, then by definition of $\arg\max_{\succsim}$, $x \succsim y$ and $y \succsim x$.

□

Claim (i) in Proposition 2 states that when the set of feasible alternatives expands, the agent is always weakly better off. This is understandable, as whatever they could choose before is still available. Claim (ii) tells us that if a $\succsim$-maximiser of a set A is also a $\succsim$-maximiser of any of its subsets. This is commonly referred to a **independence of irrelevant alternatives**. Claim (iii) is showing that if we consider a finite set, then there is always one element that is weakly preferred to every element in the set — a claim that does not necessarily hold if the set is not finite. Finally, property (iv) says not only that the agent must be indifferent between any two $\succsim$-maximisers, but also that, if the agent is indifferent between two elements, either they are both $\succsim$-maximisers or neither is.

Exercise 2. *Show that, if $B \subseteq A$, then $B \cap \arg\max_{\succsim} A \subseteq \arg\max_{\succsim} B$.*

Exercise 3. *For a finite set X and a binary relation $\succ$ on X, let the set of **maximal** elements of subset $A \subseteq X$ be defined as those for which there is no element that dominates them $\mathrm{MAX}_{\succ} A := \{x \in A \mid \nexists y \in A : y \succ x\}$.*

(i) Show that if $\succsim$ is a preference relation and $\succ$ its asymmetric part, then $\arg\max_{\succsim} A = \mathrm{MAX}_{\succ} A$ $\forall A \in 2^X$.

(ii) Now suppose that $\succsim$ is reflexive and transitive, but not necessarily complete. What is the relation between $\arg\max_{\succsim} A$ and $\mathrm{MAX}_{\succ} A$ $\forall A \in 2^X$?

*(iii) Prove that $\mathrm{MAX}_{\succ} : 2^X \to 2^X$ is a choice function if and only if $\succ$ is an **acyclic** binary relation on X, i.e., there is no sequence $x_1, x_2, \ldots, x_n \in X$ such that $x_1 \succ x_2 \succ \cdots \succ x_n \succ x_1$.*

Let us introduce the following property on choice functions due to Houthakker (1956):

Definition 4. A choice function $C : 2^X \to 2^X$ satisfies **Houthakker's Axiom of Revealed Preference** (HARP) if $\forall x, y \in X$, $\{x, y\} \subseteq A \cap B$, $x \in C(A)$ and $y \in C(B)$, then $x \in C(B)$ and $y \in C(A)$.

You will find that HARP is oftentimes called the *weak axiom of revelead preference.*

As the next result shows, if an agent always chooses their preferred elements in the feasible set, then their choices satisfy HARP. But if an agent's choices satisfy HARP, we can interpret these choices as maximising a preference relation; and, importantly, we can recover their preferences by observing their choices.

Theorem 1. *Let X be a finite set. A choice function $C : 2^X \to 2^X$ satisfies HARP if and only if there is a preference relation $\succsim \subseteq X \times X$ such that $C(A) = \arg\max_{\succsim} A \ \forall A \in 2^X$.*

Proof. $\Longrightarrow$: (only if) Define $\succsim \subseteq X \times X$ as follows: $\forall x, y \in X$, $x \succsim y$ if $\exists A \in 2^X$ such that $x, y \in A$ and $x \in C(A)$. Completeness of $\succsim$ follows from the fact that, $\forall x, y \in X$, as $C(\{x, y\})$ is nonempty and a subset of $\{x, y\}$, then $x \in C(\{x, y\}) \Longrightarrow x \succsim y$ or $y \in C(\{x, y\}) \Longrightarrow y \succsim x$.

To show transitivity, let $x, y, z \in X$ such that $x \succsim y$ and $y \succsim z$; we want to show $x \succsim z$. By definition of $\succsim$, $\exists A \ni x, y$ and $B \ni y, z$ such that $x \in C(A)$ and $y \in C(B)$. Now we want to find a set $E \ni x, z$ and show that $x \in C(E) \Longrightarrow x \succsim z$ (by definition of $\succsim$). Take $\{x, y, z\}$. If $x \in C(\{x, y, z\})$, we are done. If $y \in C(\{x, y, z\})$, as $x \in C(A)$ and $x, y \in A \cap \{x, y, z\}$, by HARP we have that $x \in C(\{x, y, z\})$ and the result follows. And if $z \in C(\{x, y, z\})$, as $y \in C(B)$ and $y, z \in B \cap \{x, y, z\}$, HARP implies that $y \in C(\{x, y, z\})$ and we are back to the previous case, where we showed that $x \in C(\{x, y, z\})$.

We then need to show that $C(A) = \arg\max_{\succsim} A$, $\forall A \in 2^X$. By definition of $\succsim$, $x \in C(A) \Longrightarrow x \succsim y \ \forall y \in A$, which, by definition of $\arg\max_{\succsim} A$ implies that $x \in \arg\max_{\succsim} A$; hence $C(A) \subseteq \arg\max_{\succsim} A$. Now, we show that $\arg\max_{\succsim} A \subseteq C(A)$ to conclude that $\arg\max_{\succsim} A = C(A)$. Take $x \in \arg\max_{\succsim} A$ ($\subseteq A$). This implies that $A \neq \emptyset$ and thus that $\exists y \in C(A)$ (as choice functions on nonempty sets are nonempty). As $x \in \arg\max_{\succsim} A$ and $y \in A$ implies that $x \succsim y$, then, by how $\succsim$ was defined, $\exists B \in 2^X$ such that $x, y \in B$ and $x \in C(B)$. As $x, y \in A \cap B$, $x \in C(B)$ and $y \in C(A)$, by HARP, $x \in C(A)$; that is, $x \in \arg\max_{\succsim} A \Longrightarrow x \in C(A)$.

$\Longleftarrow$: (if) For some preference relation $\succsim \subseteq X \times X$, define $C : 2^X \to 2^X$ such that $C(A) = \arg\max_{\succsim} A \ \forall A \in 2^X$. By definition of $\arg\max_{\succsim} A$, $C(A) \subseteq A$; and by Proposition 2(ii), $A \neq \emptyset \Longrightarrow C(A) = \arg\max_{\succsim} A \neq \emptyset$. Hence, C is a choice function.

Now we show it satisfies HARP. Take any x, y such that $\{x, y\} \subseteq A \cap B$, $x \in C(A)$, and $y \in C(B)$. As $y \in A$ and $x \in C(A) = \arg\max_{\succsim} A$, then $x \succsim y$; a symmetric argument shows that $y \succsim x$. By Proposition 2(iii), $x \sim y$ and $\{x, y\} \cap \arg\max_{\succsim} E \iff \{x, y\} \subseteq \arg\max_{\succsim} E = C(E)$, which applies to $E = A, B$; this concludes the proof. □

Another way to state HARP is by decomposing it in two properties of choice functions $C : 2^X \to 2^X$. The first is Sen's (1971) α:

Property α. If $x \in B \subseteq A \subseteq X$ and $x \in C(A)$, then $x \in C(B)$.

The intuition behind this axiom can be illustrated as follows: if you choose raspberry jam when you can choose between {raspberry, strawberry, blueberry, orange}, then you choose it too when you only {raspberry, strawberry} are available. Note that property α — which refers to the independent of irrelevant alternatives for choice functions — is the counterpart for choice functions to the analogous property we showed for $\arg\max_{\succsim}$ in Proposition 2(i).[4]

The second property is Sen's (1971) β, also called *expansion consistency*:

Property β. If $B \subseteq A \subseteq X$, $x, y \in C(B)$, and $y \in C(A)$, then $x \in C(A)$.

Exercise 4. *(i) Show that Sen's α is equivalent to the following property: if $B \subseteq A$, then $B \cap C(A) \subseteq C(B)$.*

(ii) Show that Sen's β is equivalent to the following property: if $B \subseteq A$ and $C(A) \cap C(B) \neq \emptyset$, then $C(B) \subseteq C(A)$.

(iii) Let $C : 2^X \to 2^X$ be a choice function. Prove that HARP is equivalent to Sen's α and β. Conclude on the properties that $\arg\max_{\succsim}$ satisfies, where $\arg\max_{\succsim} A := C(A)$, $\forall A \in 2^X$.

4. Preferences and Utility

We have seen in the previous section necessary and sufficient conditions to interpret an agent's choices as being driven by preference maximisation. In this section, we are going to understand in which circumstances we can think of agents' behaviour as though they are maximising a utility function.

Definition 5. A utility function $u : X \to \mathbb{R}$ represents $\succsim \subseteq X \times X$ if $x \succsim y \iff u(x) \geq u(y)$, $\forall x, y \in X$.

For $\succsim \subseteq X \times X$ and its asymmetric part $\succ$ let us define, for any subset $A \subseteq X$,

(i) $A_{\succsim x} := \{y \in A \mid y \succsim x\}$;

(ii) $A_{\succ x} := \{y \in A \mid y \succ x\}$;

(iii) $A_{x \succsim} := \{y \in A \mid x \succsim y\}$; and

(iv) $A_{x \succ} := \{y \in A \mid x \succ y\}$.

These sets can be understood as the alternatives in A that are (i) weakly preferred to x, (ii) strictly preferred to x, (iii) weakly less preferred than x, and (iv) strictly less preferred than x.

[4] While a compeling property, it is also easy to entertain situations where it may fail. For instance, if decision-makers fail to consider all possible alternatives but instead consider only a subset of the available elements, called their consideration set. This is indisputably the case: e.g., Amazon sells over 12 million items and it is unrealistic to think consumers consider all of them. For a conceptualisation of consideration sets see, e.g., Masatlioglu et al. (2012).

Proposition 3. *Let X be finite. $\succsim \subseteq X \times X$ is a preference relation if and only if it admits a utility representation u.*

Proof. The "if" part is straightforward. For the "only if" part, define $u(x) := |X_{x\succsim}|$. Note that for any $x \succsim y$, $X_{y\succsim} \subseteq X_{x\succsim}$ and therefore $u(x) \geq u(y)$. If $\neg(x \succsim y)$, by completeness we have that $y \succ x$. Transitivity yields $X_{x\succsim} \subseteq X_{y\succsim}$. But $y \in X_{y\succsim}$, as $y \succsim y$, but $y \notin X_{x\succsim}$, and therefore $X_{x\succsim} \subsetneq X_{y\succsim}$, and then $u(y) > u(x)$. □

Are utility representations unique? The answer is no: for any strictly increasing function $f : \mathbb{R} \to \mathbb{R}$, if u represents a preference relation $\succsim$ on X, then $v := f \circ u$ does too. However,

Proposition 4. *If $\succsim$ and $\hat{\succsim}$ are two different preference relations on X, then they cannot be represented by the same utility function u.*

Exercise 5. *Prove Proposition 4. Conclude that utility representations are unique up to positive monotone transformations.*

Can we go beyond finite set of alternatives?

Proposition 5. *Let X be countable. $\succsim \subseteq X \times X$ is a preference relation if and only if it admits a utility representation u.*

Proof. Again we focus on the "only if" part. Since X is countable, let us fix an order on $X = \{x_1, x_2, \ldots\}$. Because it is not necessarily finite, it can be the case that $|X_{x\succsim}| = |X_{y\succsim}| = \infty$, even if $x \succ y$ (i.e., not $y \succsim x$). Define

$$u(x) := \sum_{n \in \left\{m \mid x_m \in X_{x\succsim}\right\}} 2^{-n}.$$

As X is countable, u is well-defined as the sum is finite.
Let $x \succsim y$. Then, $X_{y\succsim} \subseteq X_{x\succsim}$ (transitivity) $\implies u(x) \geq u(y)$. If $\neg(x \succsim y)$, then $y \succ x$ and $X_{x\succsim} \subseteq X_{y\succsim} \implies u(y) \geq u(x)$. As $y = x_m$ for some finite $m \in \mathbb{N}$, then $u(y) \geq u(x) + 2^{-m} > u(x)$. □

What if X is not countable?

Example 1. The canonical example is with lexicographic preferences in $X = \mathbb{R}^2$, where the agent considers the first dimension and only in case of a tie do they resort to the second dimension: $x \succsim y$ if $x_1 > y_1$ or $x_1 = y_1$ and $x_2 \geq y_2$. While $\succsim$ is a preference relation on X it admits no utility representation. To see this, suppose it did, $u : X \to \mathbb{R}$. Then, for any $r \in \mathbb{R}$, we have that $u(r,1) > u(r,0)$ as $(r,1) \succ (r,0)$. Moreover, for any $r' > r$, $u(r',0) > u(r,1)$. Then $\{(u(r,0), u(r,1)) \mid r \in \mathbb{R}\}$ is an uncountable collection of nonempty and disjoint open intervals. However, for $r \in \mathbb{R}$, $(u(r,0), u(r,1))$ is a nonempty open interval, and as $\mathbb{Q}$ is dense in $\mathbb{R}$,[5] we

[5] That is, for any $x \in \mathbb{R}$ and any $\epsilon > 0$, $B_\epsilon(x) \cap \mathbb{Q} \neq \emptyset$.

can find a rational number $q \in (u(r,0), u(r,1))$. As the set of rational numbers is countable, we obtain a contradiction.

The main intuition for why a utility representation is not possible in Example 1 is that there are 'too many' "indifference sets": every point in $\mathbb{R}^2$ is an indifference set and we want to represent every indifference set with a real number.

Definition 6. Let $\succsim \subseteq X \times X$. A subset $X^* \subseteq X$ is **order-dense** in X if for every $x, y \in X : x \succ y$, there is $z \in X^*$ such that $x \succsim z \succ y$.

As $\mathbb{R}$ has a countable $\geq$-dense subset — e.g., that of the rational numbers — we preferences should be well captured by a countable number of "indifference sets." The next result shows that this is an if and only if condition:

Theorem 2. *$\succsim \subseteq X \times X$ is a preference relation and there is a countable order-dense $X^* \subseteq X$ if and only if $\succsim$ admits a utility representation.*

Proof. $\implies$: (only if)
Fix an order on $X^* = \{x_1^*, x_2^*, \ldots\}$. Define

$$u(x) := \sum_{n \in \left\{m \mid x_m \in X_{x\succsim} \cap X^*\right\}} 2^{-n}.$$

As X^* is countable, u is well-defined as the sum is finite.
Let $x \succsim y$. Then, $X_{y\succsim} \subseteq X_{x\succsim}$ (transitivity) $\implies X_{y\succsim} \cap X^* \subseteq X_{x\succsim} \cap X^* \implies u(x) \geq u(y)$. If $\neg(x \succsim y)$, then $y \succsim x$ (completeness) $\implies X_{x\succsim} \cap X^* \subseteq X_{y\succsim} \cap X^*$. As $\neg(x \succsim y)$ and $y \succsim x$, $y \succ x$ and, as X^* is order-dense in X, there is $x_m^* \in X_{y\succsim} \cap X^*$ and $x_m^* \notin X_{x\succsim} \cap X^*$. We then conclude $u(y) \geq u(x) + 2^{-m} > u(x)$.

$\impliedby$: (if)
Let $u : X \to \mathbb{R}$ be a utility representation of $\succsim$: $u(x) \geq u(y) \iff x \succsim y$. That $\succsim$ is complete and transitive is straightforward to verify. Then, let us construct our countable, order-dense $X^* \subseteq X$.
Let $u(X) := \{u(x) \in \mathbb{R} \mid x \in X\}$.
For every $(p,q) \in \mathbb{Q}^2$ such that $p < q$ and $(p,q) \cap u(X) \neq \emptyset$, take one $x_{p,q} \in X$ such that $u(x_{p,q}) \in (p,q)$, and let $X_{p,q} := \{x_{p,q}\}$.
And for every $p \in \mathbb{Q}$ such that $\exists x \in X : u(x) = \inf([p,\infty) \cap u(X))$, take one x_p such that $u(x_p) = \inf([p,\infty) \cap u(X))$, and define $X_p := \{x_p\}$.
By construction, $\cup_{(p,q)\in\mathbb{Q}^2 : p<q} X_{p,q}$ and $\cup_{p\in\mathbb{Q}} X_p$ are countable subsets of X and therefore so is $X^* := \left(\cup_{p\in\mathbb{Q}} X_p\right) \cup \left(\cup_{(p,q)\in\mathbb{Q} \mid p<q} X_{p,q}\right)$. To see that X^* is order-dense in X take any $x, y \in X$

such that $x \succ y$. If $\exists z \in X : x \succ z \succ y \iff u(x) > u(z) > u(y)$, then

$$\begin{aligned} u(x) > u(z) > u(y) &\implies \exists p,q \in \mathbb{Q} : u(x) \geq q \geq u(z) \geq p > u(y), \quad \text{and } p < q \\ &\implies (p,q) \cap u(X) \neq \emptyset \\ &\implies \exists x_{p,q} \in X^* \subseteq X : u(x) \geq u(x_{p,q}) > u(y) \\ &\implies x \succsim x_{p,q} \succ y. \end{aligned}$$

If $\nexists z \in X : x \succ z \succ y$, then there is $p \in \mathbb{Q} : u(x) > p > u(y)$. Moreover, as $u(x) = \inf([p,\infty) \cap u(X))$, $\exists x_p \in X^* : u(x_p) = u(x)$. Hence, $u(x) = u(x_p) > u(y) \iff x \succsim x_p \succ y$. □

Note that Theorem 2 subsumes the previous utility-representation result, Proposition 5, as, for any preference relation $\succsim$ on countable X, X is already an order-dense subset of itself.

In the next exercise, we will try to see how restrictive our model is by considering procedures other than utility maximisation.

Exercise 6. *A consumer is choosing between books from a finite set X. They have a utility function, $u : X \to \mathbb{R}$, and a 'threshold utility' $\bar{u}$. The bookseller sets the books in a given fixed ordering S which is complete, transitive, and antisymmetric, e.g., alphabetically by title (assuming no two books have the same title). Then, in any set of books $A \subseteq X$ in display, the consumer starts searching according to S in decreasing order (e.g., alphabetically), and chooses the first book for which the utility is equal or exceeds $\bar{u}$. If there is no such book in A, then they just go with the one with the highest utility.*

(i) Does this procedure satisfy α, β, both, or neither?

(ii) Let $\succsim$ be such that $x \succsim y$ if and only if $u(x) \geq u(y)$. Can the bookseller learn the consumer's preferences $\succsim$? If so, how? If not, why?

(iii) Discuss the statement: If choices are consistent with HARP, we are sure that consumers are choosing their most preferred items.

4.1. Choice Theory and Optimisation

To conclude this section, note that the results we proved earlier allow us to derive a number of useful properties for optimisation without needing to know much about the function or set over which we are optimising.

Let $f : X \to \mathbb{R}$ be a real-valued function on X. Define, for every $A \in 2^X$,

$$\max_{x \in A} f(x) := \{f(x) \mid x \in A \text{ and } f(x) \geq f(y), \forall y \in A\};$$
$$\arg\max_{x \in A} f(x) := \{x \in A \mid f(x) \geq f(y), \forall y \in A\};$$
$$\min_{x \in A} f(x) := \{f(x) \mid x \in A \text{ and } f(x) \leq f(y), \forall y \in A\};$$
$$\arg\min_{x \in A} f(x) := \{x \in A \mid f(x) \leq f(y), \forall y \in A\}.$$

Note that $\min_{x \in A} f(x) = -\max_{x \in A} -f(x)$ (why?).

Then, reusing the results for preference relations we can deduce the following:

Proposition 6. *The following properties hold:*

(i) If $B \subseteq A \subseteq X$, then for any $x \in \arg\max_{z \in A} f(z)$ and $y \in \arg\max_{z \in B} f(z)$, $f(x) \geq f(y)$.

(ii) For any nonempty $A \subseteq X$ and X is finite, $\arg\max_{x \in A} f(x) \neq \emptyset$.

(iii) For $x, y \in A \subseteq X$, $f(x) = f(y)$ and $\{x, y\} \cap \arg\max_{z \in A} f(z) \neq \emptyset$ if and only if $\{x, y\} \subseteq \arg\max_{z \in A} f(z)$.

(iv) If $x \in B \subseteq A \subseteq X$, and $x \in \arg\max_{z \in A} f(z)$, then $x \in \arg\max_{z \in B} f(z)$.

Exercise 7. *Prove Proposition 6 by making use of the results derived above.*

5. Limited Observability (*)

Suppose that you want to test whether an agent's choice function admit a preference representation. Technically speaking, you would need to observe a mapping $C : 2^X \to 2^X$. This is a lot of data: if $|X| = 20$ we need to observe choices from over 1 million different subsets of X.

On the other hand, consider the following example:

Example 2. Suppose that $X = \{x, y, z\}$ and you only observe $C(\{x, y\}) = \{x\}$, $C(\{y, z\}) = \{y\}$, and $C(\{x, z\}) = \{z\}$. HARP (as well as Sen's α and β) is trivially satisfied, but no preference relation $\succsim$ exists that is consistent with $C(A) = \arg\max_{\succsim} A$ for $A \in \{\{x, y\}, \{y, z\}, \{x, z\}\}$.

We need to, somehow, generalize HARP. In particular, we want to be able to infer that if the data tells us that $x \succ y$ and $y \succ z$, then we should infer that $x \succ z$. For that purpose we first need the following definition:

Definition 7. Let $\succsim$ be a binary relation on X. $T(\succsim)$ is the **transitive closure** of $\succsim$ if (i) $T(\succsim)$ is a transitive binary relation on X, (ii) $\succsim \subseteq T(\succsim)$, that is, $x \succsim y \implies xT(\succsim)y$, (iii) any binary relation $\hat{\succsim} \subsetneq T(\succsim)$ is either intransitive or $\neg(\succsim \subseteq \hat{\succsim})$.

We can obtain the transitive closure when X is finite as follows: First, for any binary relation on X, $\succsim$, let $f(\succsim)$ such that if $x \succsim y$ and $y \succsim z$, then $xf(\succsim)z$. Define $\succsim_1 := f(\succsim)$ and, for $n > 1$,

$\succsim_n := f(\succsim_{n-1})$. Then, $\succsim_{|X|} = T(\succsim)$. The next theorem ensures that $T(\succsim)$ is well-defined:

Theorem 3. *For any $\succsim$ binary relation on X, $\exists T(\succsim)$.*

We will see the proof for it later on in the course.

Now that we know that we can render a binary relation transitive, we need to make it complete.

It is easy to complete a binary relation $R \subseteq X^2$: for any $x, y \in X$ such that $\neg(x \succsim y), \neg(y \succsim x)$, we could simply add either (x, y) or (y, x) or both to the original binary relation R. Following this *completion* procedure we would indeed end up with a complete binary relation that contains R. However, we will need to be a bit more careful, as we need to complete the binary relation in a way such that the resulting binary relation is transitive *and* its strict part contains the strict part of the original binary relation R. That is, a completion is not enough.

For our purposes, we want to rely on the concept of an extension:

Definition 8. Let $\succsim$ be a preorder on X. An **extension** of $\succsim$ is a complete preorder[6] $\trianglerighteq$ on X such that $\succsim \subseteq \trianglerighteq$ and $\succ \subseteq \triangleright$, where $\succ$ and $\triangleright$ are the asymmetric parts of $\succsim$ and $\trianglerighteq$, respectively.

The following is a version of Sziplrajn's theorem that says that an extension always exists:

Theorem 4. ***(Sziplrajn)*** *For any nonempty set X and preorder $\succsim$ on X, there is an extension of $\succsim$.*

Let us pursue our intuition that we should be able to infer preferences about two elements even when they are never available at the same time. First, given a subset $Y \subseteq 2^X$ and a choice function $C : Y \to Y$, let us define a binary relation R^D on X such that xR^Dy if there is $A \in Y$ for which $y \in A$ and $x \in C(A)$, in which case we say that x is **directly revealed preferred to** y. We then define the revealed preference relation R as the transitive closure of R^D, that is $R := T(R^D)$, and we say that x is **revealed preferred to** y and if $x \succsim y$.

Finally, we need to preserve strict preferences. If we were to define the revealed preference relation from Example 2, we would get that the agent would be indifferent with respect to any element. Instead, we want a restrictive interpretation of the data. This is given by the concept of revealed strict preference: Given the same choice function on Y, we say that x is **revealed strictly preferred to** y — and write xSy — if there is $A \in Y$ such that $y \in A \setminus C(A)$ and $x \in C(A)$. That is, if x was chosen and y was not chosen but could have been chosen, then we understand that it cannot be the case that y is weakly preferred to x.

Definition 9. Let $Y \subseteq 2^X$ and let $C : Y \to 2^X$ be a choice function. C satisfies the **Generalized Axiom of Revealed Preference** (GARP) if $\nexists x, y \in X$ such that x is revealed preferred to y and y is revealed strictly preferred to x.

[6] That is, a preference relation.

The main result of this section is the following:

Theorem 5. *Let $Y \subseteq 2^X$. A choice function $C : Y \to 2^X$ satisfies GARP if and only if there is a preference relation $\succsim \subseteq X^2$ such that $C(A) = \arg\max_{\succsim} A$ for any $A \in Y$.*

Proof. The 'if' part is straightforward to show; we focus on the 'only if' part. By Theorem 3, R is well defined. By GARP, S is a subset of the asymmetric part of R. Note that $\tilde{R} := R \cup \{(a,a) \mid a \in X\}$ is a preorder on X. Let $\succsim$ be an extension of $\tilde{R}$ such that $\succsim$ is a complete preference relation on X; by Theorem 4, $\succsim$ exists. By definition of an extension, the result follows: $C(A) = \arg\max_{\succsim} A$ for any $A \in Y$. □

While revealed preference is a powerful way to model behaviour, as it enable us to use optimisation to describe behaviour, Exercise 6 recommends caution when using behaviour that is consistent with "preference maximisation" to make inferences about how well-off an agent is.

Exercise 8. *Suppose that the decision-maker has a preference relation $\succsim$ on X and their choices at any subset $A \in 2^X$ are given by $C(A) := \arg\max_{\succsim} A$.*

If, instead of observing a dataset $(A_t, C(A_t))_t$, we observe $(A_t, x_t)_t$, where $x_t \in C(A_t)$, what can only say about $\succsim$?

(Extra) Suppose $X = \{x_n\}_{n \in [10]}$, where each x_n represents an ice-cream flavor. You observe the following data:

A	$C(A)$
$\{x_5, x_7\}$	$\{x_5\}$
$\{x_1, x_7\}$	$\{x_7\}$
$\{x_4, x_8, x_{10}\}$	$\{x_4, x_{10}\}$
$\{x_1, x_2, x_3, x_6\}$	$\{x_1, x_3, x_6\}$
$\{x_3, x_9, x_{10}\}$	$\{x_3, x_{10}\}$
$\{x_2, x_8, x_9\}$	$\{x_8, x_9\}$

Write a program (in Python/Julia/R) to test whether the dataset satisfies GARP and, if so, provide a preference relation $\succsim \in X \times X$ such that $C(A) = \arg\max_{\succsim} A$.

Suggestion: To derive preference relations, when the dataset satisfies GARP, create a $|X| \times |X|$ matrix M of zeros and replace the ij-th coordinate whenever $x_j \in A$ and $x_i \in C(A)$. Then, obtain the transitive closure of M (which you can do easily with matrix multiplication).

6. Further Reading

Standard References: Mas-Colell et al. (1995, Chapters 1, 3A-B), Rubinstein (2018, Chapter 1, 3), Kreps (2012, Chapter 1), Kreps (1988, Chapters 1-3).

Background on Order Theory: Ok (2007, Chapter A1).

Related questions/topics: representation and interpretation of incomplete preferences; reference points and consideration sets; identifying inference based on choice; social choice; search, satisficing, choice from lists, and framing effects; similarity.

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