ECON0106: **Microeconomics**

13. Monotone Comparative Statics in Games[*]

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1. Overview

This lecture discusses monotone comparative statics of fixed points. This is a tool that you can use for research in a variety of topics and fields.

Here are some examples of questions that have been addressed using these tools:

– (Macro) Comparative statics on equilibrium prices and quantities when there is a demand shock induced by a change in consumers' preferences (e.g. Acemoglu and Jensen, 2015).

– (Econometrics) Nonparametric partial identification of treatment response with social interactions (e.g. Lazzati 2015, with an application to studying the effect of police per capita on crime rates).

– (Health) Empirical antitrust implications of centralised matching systems on wages of medical residents (e.g. Agarwal, 2015).[1]

– (Education) The empirical consequences of affirmative action in university admission (e.g. Dur et al., 2020; Aygün and Bó, 2021).

Needless to say that theorems on fixed points and comparative statics are the bread and butter of theoretical research.

2. Ordering Sets – Again

Recall the notion of **strong set order** $\geq_{ss}$ (Topkis, 1979, 1998; Milgrom and Shannon, 1994), where $\geq_{ss}$ is a binary relation on 2^X for some partially ordered set $(X, \geq)$:

Definition 1. We say that S' **strong set dominates** S (writing $S' \geq_{ss} S$) if $\forall x' \in S', x \in S$, $x \vee x' \in S'$ and $x \wedge x' \in S$.

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[1]Fun fact: Nobel Prize winner, Alvin Roth not only made fundamental contributions to the theory of market design leveraging on lattice theory, these contributions then underlied crucial reforms in the US national residency matching program and organ donation programs, which he helped redesign. See https://www.nobelprize.org/uploads/2018/06/advanced-economicsciences2012.pdf.

As mentioned previously, this ordering of sets can be too demanding and therefore inapplicable to many situations.

One case where the strong set order is often unreasonable is when we want to compare sets of fixed points when some fundamental changes. For instance, suppose that you have a set of players I and each player i can choose strategies s_i in S_i. Their choices — characterised by some model, not necessarily best responses — are given by a mapping $B_i : S_{-i} \times \Theta_i \rightrightarrows S_i$ that depends on their opponents' choices and some parameter θ_i. You characterise the set of fixed points $\mathscr{F}(B,\theta)$, that is, the set of choices that are consistent, $s_i \in B_i(s_{-i},\theta_i)$ for every $i \in I$, and how they depend on θ. One major issue is that is hardly ever going to be the case that the sets of equilibria are strong-set ordered. This is because if s and s' are equilibrium strategy profiles in two different games, it is going to be extremely difficult to have that $s \vee s'$ and $s \wedge s'$ be equilibrium strategy profiles in either. So, to start with, we need a less stringent way of ordering sets. This is one possible motivation for the weak set order (Che et al., 2021).

Definition 2. We say that

(i) S' **upper weak set dominates** S (writing $S' \geq_{uws} S$) if $\forall x \in S$, $\exists x' \in S'$ such that $x' \geq x$;

(ii) S' **lower weak set dominates** S ($S' \geq_{lws} S$) if $\forall x' \in S'$, $\exists x \in S$ such that $x' \geq x$;

(iii) S' **weak set dominates** S ($S' \geq_{ws} S$) if S' both upper and weak set dominates S; that is, $\geq_{ws} = \geq_{uws} \cap \geq_{lws}$.

How do these two set orders compare?

Lemma 1. *(i) The strong set order is transitive and antisymmetric on non-empty sets. It is not necessarily either reflexive or irreflexive.*

(ii) The weak set order is transitive and reflexive,[2] *but not necessarily antisymmetric.*

(iii) For all nonempty subsets $S,T \subseteq X$, $S \geq_{ss} T \implies S \geq_{ws} T$.

(iv) The strong set order is closed under intersection, i.e. for all non-empty $S,S',T,T' \subseteq X$ *such that* $S' \geq_{ss} S$ *and* $T' \geq_{ss} T$, $S' \cap T' \geq_{ss} S \cap T$. *It is not necessarily closed under union.*

(v) The weak set order is closed under union, i.e. for all non-empty $S,S',T,T' \subseteq X$ *such that* $S' \geq_{ws} S$ *and* $T' \geq_{ws} T$, $S' \cup T' \geq_{ws} S \cup T$. *It is not necessarily closed under intersection.*

Exercise 1. *Prove Lemma 1.*

Exercise 2. *Assume the conditions in Exercise 1 in Lecture Note 4. Provide necessary and sufficient conditions on* $w', w \geq 0, p', p \in \mathbb{R}^k_{++}$ *so that* $B(p',w') \geq_{ws} B(p,w)$.

[2] Hence, a preorder.

3. Fixed-Point Theorems

In this section we'll discuss two fixed-point theorems that are based on the strong and weak set orders. We will make use of them to show existence of equilibria later on when we look at comparative statics.

3.1. Tarski and Zhou Fixed-Point Theorems

Let X be a lattice.[3]

Definition 3. A function $f : X \to X$ is said to be **monotone** if it is order-preserving, i.e. $x \geq y \implies f(x) \geq f(y)$. A correspondence $F : X \rightrightarrows X$ is said to be **monotone** if $x \geq y \implies F(x) \geq_{ss} F(y)$.

Let $\mathscr{F}(F) := \{x \in X \mid x \in F(x)\}$ denote the set of **fixed points** of a self-correspondence F on X; we will abuse notation and write as well $\mathscr{F}(f) := \{x \in X \mid x = f(x)\}$ to denote the set of fixed points of a self-map f on X.

Tarski's (1955) fixed-point theorem pertains to functions and was later generalised to correspondences by Zhou (1994). It can be stated as follows:

Theorem 1. *(Tarski, 1955) Let X be a complete lattice and f be a self-map on X. If f is monotone, then $\mathscr{F}(f)$ is a non-empty complete lattice.*

We will prove a more humble statement:

Lemma 2. *Let X be a complete lattice and f be a self-map on X. If f is monotone, then $\mathscr{F}(f)$ is nonempty and has a largest element.*

Proof. Let $S := \{x \in X : f(x) \geq x\}$. As X is a complete lattice $f(\inf_X X) \geq \inf_X X$, and therefore $S \neq \emptyset$. As $S \subseteq X$ and X is a complete lattice, $y := \sup_X S \in X$. Then, for any $x \in S$,

$$
\begin{aligned}
y \geq x &\implies f(y) \geq f(x) \geq x && \text{as } f \text{ is monotone and } x \in S \\
&\implies f(y) \geq y && \text{as } f(y) \text{ is an upper bound of } S \text{ and } y = \sup_X S \\
&\implies f(f(y)) \geq f(y) && \text{as } f \text{ is monotone} \\
&\implies f(y) \in S \\
&\implies y := \sup_X S \geq f(y) \\
&\implies y = f(y) && \text{by antisymmetry.}
\end{aligned}
$$

□

We provide a heuristic proof in the Appendix A to the lecture notes for completeness, but you

[3]We omit the dependence on $\geq$ to not overburden the text.

don't need to know it; check it at your own risk.

Zhou (1994) then generalised the theorem to monotone correspondences:

Theorem 2. *(Zhou 1994, Theorem 1) Let X be a complete lattice and $F : X \rightrightarrows X$ be nonempty-valued. If F is monotone and, $\forall x \in X$, $F(x)$ is a complete sublattice, then $(\mathscr{F}(F), \geq)$ is nonempty complete lattice.*

Again, we leave the details to the Appendix B, but we state a counterpart to Lemma 2 that requires only that very lemma (which we showed above) to prove it.

Lemma 3. *Let X be a complete lattice and $F : X \rightrightarrows X$ be nonempty-valued. If F is monotone and, $\forall x \in X$, $F(x)$ is a complete sublattice, then $(\mathscr{F}(F), \geq)$ is nonempty and has a largest element.*

Exercise 3. *Prove Lemma 3 by relying only on Lemma 2.*

3.2. Li–Che–Kim–Kojima Fixed Point Theorem

Let $F : X \rightrightarrows Y$, where X, Y are partially ordered sets. For $S \in 2^X$, we write $F(S) := \cup_{x \in S} F(x)$.

Definition 4. F is said to be

(i) **upper weak set monotone** if $F(x') \geq_{uws} F(x)$ $\forall x' \geq x$;

(ii) **lower weak set monotone** if $F(x') \geq_{lws} F(x)$ $\forall x' \geq x$;

(iii) **weak set monotone** if $F(x') \geq_{ws} F(x)$ $\forall x' \geq x$;

(iv) **strong set monotone** if $F(x') \geq_{ss} F(x)$ $\forall x' \geq x$.

Lemma 4. *(Che et al., 2021, Lemma 2) Let $F : X \rightrightarrows Y$, where X, Y are partially ordered sets. If F is weak set monotone, then for any subsets $S', S \subseteq X$ such that $S' \geq_{ws} S$, $F(S') \geq_{ws} F(S)$.*

In the sequel we will say that a set X is a *compact partially ordered metric space*; you can read "compact subset a Euclidean space" ($\mathbb{R}^k$).[4] In fact, as in applications it will be often assumed that X is a subset of a Euclidean space, it may be convenient to know that in this case X being a compact lattice is equivalent to it being a complete lattice.[5,6]

Theorem 3. *(Li 2014; Che et al. 2021, Theorem 6) Let X be a compact partially ordered metric space. Let $F : X \rightrightarrows X$ be a nonempty- and closed-valued correspondence on X. If F is upper (resp. lower) weak set monotone and $\exists x, y \in X$ such that $x \leq y \in F(x)$ (resp. $x \geq y \in F(x)$), then it admits a maximal (resp. minimal) fixed point.*

[4] More rigorously, we use this to mean that it is endowed with the metric and natural topology induced by its partial order ($\geq$), and then we assume it is compact with respect to this topology.

[5] Note that we are saying a complete lattice, not a complete *sub*lattice of $\mathbb{R}^k$! An example: $\{(0,0),(1,0),(0,1),(2,2)\}$ is a complete lattice, a sublattice of $\mathbb{R}^2$, but not a complete *sub*lattice of $\mathbb{R}^2$.

[6] The observation generalises more broadly, in that a complete lattice will, in many domains of interest, imply it is also compact; see Che et al. (2021, Theorem S2).

Proving the result goes far beyond the scope of this class; for those who are interested, a brief sketch of the steps is in the Appendix C.

4. Monotone Comparative Statics on Fixed Points

Villas-Boas (1997) proved weak monotone comparative statics results for functions that are very general. The first considers decreasing functions:

Theorem 4. *((Villas-Boas, 1997, Theorem 3)) Let $(X,\geq)$ be a preordered set, and $f,g: X \to X$. If (i) $f \gg g$, (ii) $\forall x, y \in X: x \geq y \implies f(y) \geq f(x)$, then $\forall x \in \mathscr{F}(f), y \in \mathscr{F}(g)$, $\neg(y > x)$.*

Proof. Suppose not, i.e., $\exists x \in \mathscr{F}(f), y \in \mathscr{F}(g): y > x$. Then

(i) $f(y) > g(y) \because f \gg g$;

(ii) $g(y) = y \because y \in \mathscr{F}(g)$;

(iii) $y > x$, by assumption;

(iv) $x = f(x) \because x \in \mathscr{F}(f)$; and

(v) $f(x) \geq f(y) \because y > x \implies y \geq x \implies f(x) \geq f(y)$;

a contradiction. □

The next theorem considers increasing functions:

Theorem 5. *(Villas-Boas, 1997, Theorems 4 and 5) Let X be a partially ordered set, and $f,g: X \to X$.*

(i) If (i) $\forall x \in X$, $X_{\geq x}$ is a complete lattice, and (ii) f is weakly increasing, then for every fixed point of g, $x \in \mathscr{F}(g)$ such that $f(x) \geq (>) x$, there is a fixed point of f $y \in \mathscr{F}(f)$ for which $y \geq (>) x$. If in addition $f \geq (>) g$, then $\forall x \in \mathscr{F}(g)$, $\exists y \in \mathscr{F}(f)$ such that $y \geq (>) x$.

(ii) If (i) $\forall x \in X$, $X_{x\leq}$ is a complete lattice, and (ii) g is weakly increasing, then for every fixed point of f, $y \in \mathscr{F}(f)$ such that $y \geq (>) g(y)$, there is a fixed point of g $x \in \mathscr{F}(g)$ for which $y \geq (>) x$. If in addition $f \geq (>) g$, then $\forall y \in \mathscr{F}(f)$, $\exists x \in \mathscr{F}(g)$ such that $y \geq (>) x$.

Proof. We will prove (i); the proof for (ii) is symmetric.
Let $x^* \in \mathscr{F}(g): f(x^*) \geq (>) x^*$. By monotonicity, $f(x) \geq (>) x^*$ $\forall x \in X_{\geq x^*}$. Let $\tilde{f}: X_{\geq x^*} \to X_{\geq x^*}$, where $\tilde{f}(x) = f(x)$. Now apply Tarki's fixed point theorem (for (i), Lemma 2 is enough) and conclude that $\exists y \in X_{\geq x^*}$ such that $f(y) = y \geq (>) x^*$. □

As we will see in the next section, Theorem 5 pairs very well with the strong set order to obtain comparative statics for equilibria. In addition, Villas-Boas (1997) provides extensions for

Banach spaces and correspondences under very general conditions, which may prove useful if you are doing some complicated functional optimisation (as is sometimes the case in macro).

Che et al. (2021, Theorem 7) provide results for correspondences that are adjusted to the weak set order.

Theorem 6. *(Che et al., 2021, Theorem 7) Let X be a compact partially ordered metric space and $F, G : X \rightrightarrows X$.*

(i) If $\mathscr{F}(F) \neq \emptyset$, G is upper weak set monotone, nonempty- and closed-valued,
and $G(x)$ upper weak set dominates $F(x)$ for every $x \in X$,
then the set of fixed points of G upper weak set dominate those of F.

(ii) If $\mathscr{F}(G) \neq \emptyset$, F is lower weak set monotone, nonempty- and closed-valued,
and $G(x)$ lower weak set dominates $F(x)$ for every $x \in X$,
then the set of fixed points of G lower weak set dominate those of F.

Proof. Take any $x^* \in \mathscr{F}(F)$. For any $S \subseteq X$, define $S_{\geq x^*} := \{x \in S \mid x \geq x^*\}$ and $S_+(F) := \{x \in S \mid \exists y \geq x$ such that $y \in F(x)\}$ ($S_-(F)$ is analogously defined).
Let $\tilde{G}$ be a self-correspondence on $X_{\geq x^*}$ such that $\tilde{G}(x) := G(x)_{\geq x^*}$ for any $x \in X_{\geq x^*}$, that is, all the elements in $G(x)$ that dominate x^*.
We will prove that $\tilde{G}$ verifies the conditions for it to have a fixed point in $X_{\geq x^*}$, thereby showing upper weak set dominance.

(i) First, we want to show that $X_{\geq x^*}$ is a compact lattice. As any S closed, $S_{\geq x^*}$ is also closed, and as X is compact metric space, then $S_{\geq x^*}$ is compact. Hence, $X_{\geq x^*}$ is compact.

(ii) Note that $x^* \in X_+(\tilde{G}) \subseteq X_{\geq x^*}$. This is because as $x^* \in F(x^*) \leq_{uws} G(x^*)$ and therefore $\exists y \in G(x^*)$ such that $y \geq x^*$, i.e. $y \in \tilde{G}(x^*) = G(x^*) \cap X_{\geq x^*}$.
Closed-valuedness of $\tilde{G}$ follows from the facts that G is closed-valued and $X_{\geq x^*}$ is closed.

(iii) We show now that $\tilde{G}$ is nonempty-valued. As for all $x \in X_{\geq x^*}$, $G(x) \geq_{uws} F(x^*) \ni x^*$, we have for all $x \in X_{\geq x^*}$, there is $y \in G(x)$ such that $y \geq x^*$; this in turn implies that any such $y \in \tilde{G}(x) = G(x) \cap X_{\geq x^*}$, and therefore $\tilde{G}$ is nonempty-valued.

(iv) Let us now show that $\tilde{G}$ is upper weak set monotone. $\forall x, x' \in X_{\geq x^*}$, such that $x' \geq x$, and any $y \in \tilde{G}(x) \subseteq G(x)$, $\exists y' \in G(x')$ such that $y' \geq y$, as G is uws monotone. As $y \geq x^*$, then $y' \geq x^* \implies y' \in G(x') \cap X_{\geq x^*} = \tilde{G}(x')$.

(v) Therefore, the conditions for $\tilde{G}$ to have a fixed point as per Theorem 3 are satisfied, and $\exists y^* \in \mathscr{F}(\tilde{G}) \subseteq \mathscr{F}(G)$ such that $y^* \geq x^*$. We conclude $\mathscr{F}(G) \geq_{uws} \mathscr{F}(F)$.

The proof for (ii) is symmetric. □

Corollary 1. *Let X be a compact partially ordered metric space and $F, G : X \rightrightarrows X$. If F and G*

are nonempty- and closed-valued, F is lower weak set monotone, G is upper weak set monotone, and $G(x) \geq_{ws} F(x)$ for every $x \in X$, then $\mathscr{F}(G) \geq_{ws} \mathscr{F}(F)$.

5. Games with Strategic Complementarities

In this section, we'll define games is a reduced-form manner, so as to obtain results that are applicable more generally. We call a **reduced-form game** G a tuple $G = \langle I, X, B \rangle$, where (i) I is a finite set of players, $X = \times_{i \in I} X_i$ with X_i denoting player i's strategy space, and $B = (B_i)_{i \in I}$ with $B_i : X_{-i} \rightrightarrows X_i$ being a correspondence that characterises player i's behavior. We will also abuse notation and write $B(x) : X \rightrightarrows X$, such that $B(x) := \times_{i \in I} B_i(x_{-i})$, where the meaning is clear from the context. As B summarises all the components of our reduced-form game[7] we will denote the set of fixed points of G by $\mathscr{F}(B)$, defined in the same way as before, $\mathscr{F}(B) := \{x \in X \mid x_i \in B_i(x_{-i})\}$.

Our goal is provide general results that say something like "if B_i increases, then the set of equilibria increases," in a well-defined sense. For instance, if payoffs for a player's action increase, when can we say that the player will choose it more often? Another example: suppose that the government decides to regulate prices for a specific good and imposes a price cap (e.g. EU price regulation on pharmaceutical products). How will that change pricing strategies of firms unaffected by the price cap? A similar question arises with quantity quotas — think about an oil cartel.

Note that B_i may or may not be given as the best-response correspondence in a game, i.e. $B_i(x_{-i}) = \arg\max_{x_i \in X_i} u_i(x_i, x_{-i})$. This way of defining things has the virtue of being applicable to equilibrium models and solution concepts other than Nash equilibrium. After discussing the results in terms of the properties of B_i that we need, we note how to get them from results we already saw when discussing monotone comparative statics of individual choices.

Strong Complementarities. First let's use Zhou's fixed point theorem to show an equilibrium exists:

Theorem 7. *Let $X := \times_{i \in I}$ be such that, $\forall i \in I$, X_i is a complete lattice. If $B_i : X_{-i} \rightrightarrows X_i$ is strong set monotone and nonempty- and complete-sublattice-valued for every $i \in I$, then the set of fixed points of B, $\mathscr{F}(B)$, is a complete lattice.*

Proof. As B_i is complete-sublattice-valued for every $i \in I$, $B(x)$ is a complete sublattice of X (in the product order). As B_i are monotone in the strong set order, then so is B. By Zhou's fixed point theorem (Theorem 2), the set of fixed points of B is a complete sublattice and therefore has a largest and smallest element. □

[7] Note that the definition of B depends on I and X

Now let's obtain the monotone comparative statics on the sets of equilibria:

Theorem 8. *Let $X := \times_{i\in I} X_i, \tilde{X} := \times_{i\in I} \tilde{X}_i$ be such that, $\forall i \in I$, $X_i, \tilde{X}_i$ are complete lattices with respect to the same partial order, and $\tilde{X}_i \geq_{ss} X_i$. Let $B_i : X_{-i} \rightrightarrows X_i$ and $\tilde{B}_i : \tilde{X}_{-i} \rightrightarrows \tilde{X}_i$ be strong set monotone and nonempty- and complete-sublattice-valued for every $i \in I$. If $\tilde{B}_i(\tilde{x}_{-i}) \geq_{ss} B_i(x_{-i})$ for every $i \in I$ and $x_{-i} \in X_{-i}, \tilde{x}_{-i} \in \tilde{X}_{-i}$ such that $\tilde{x}_{-i} \geq x_{-i}$, then $\sup_{\mathscr{F}(\tilde{B})} \mathscr{F}(\tilde{B}) \geq \sup_{\mathscr{F}(B)} \mathscr{F}(B)$ and $\inf_{\mathscr{F}(\tilde{B})} \mathscr{F}(\tilde{B}) \geq \inf_{\mathscr{F}(B)} \mathscr{F}(B)$.*

Note that if the above result implies that (but is not equivalent to) $\mathscr{F}(\tilde{B})$ *weak set dominates* $\mathscr{F}(B)$.

Proof. As $\mathscr{F}(\tilde{B}), \mathscr{F}(B)$ are lattices, their largest and smallest elements exist. We show that the largest fixed point of $\tilde{B}$ is greater than the largest fixed point of B; the proof for the smallest is symmetric.

Let $b_i^*(x_{-i}) := \sup_{X_i} B_i(x_{-i})$ and $b_{i*}(x_{-i}) := \inf_{X_i} B_i(x_{-i})$. As B_i is complete-sublattice-valued, $b_i^*(x_{-i}), b_{i*}(x_{-i}) \in B_i(x_{-i})$. Let $b^*(x) := \sup_X B(x) = (b_i^*(x_{-i}))_{i\in I}$ and symmetrically for $b_*(x)$. Define $\tilde{b}^*$ and $\tilde{b}_*$ analogously, on $\tilde{X}$.

Claim: The largest (smallest) fixed point of b^* (b_*) is the largest (smallest) fixed point of B. The proof of this claim is left as an exercise.

Claim: $\tilde{b}^*(\tilde{x}) \geq b^*(x)$ for any $\tilde{x} \geq x$.
To see this, note that $\tilde{b}^*(\tilde{x}) \in \tilde{B}(\tilde{x})$ and $b^*(x) \in B(x)$. As $\tilde{B}_i(\tilde{x}_{-i}) \geq_{ss} B_i(x_{-i})$ for every $i \in I$ and $x_{-i} \in X_{-i}, \tilde{x}_{-i} \in \tilde{X}_{-i}$ such that $\tilde{x}_{-i} \geq x_{-i}$, then $\tilde{B}(\tilde{x}) \geq_{ss} B(x)$ for $\tilde{x} \geq x$.[8] Hence, $\tilde{x} \vee x \in \tilde{B}(\tilde{x})$ and then $x \leq \tilde{x} \vee x \leq \tilde{b}^*(\tilde{x}) = \sup_{\tilde{X}} \tilde{B}(\tilde{x})$.

Claim: $\tilde{b}^*$ is monotone.
As $\tilde{B}_i$ is strong set monotone and complete-sublattice-valued for each i, so is $\tilde{B}$. Completeness yields $\sup_{\tilde{X}} \tilde{B}(x) \in \tilde{B}(x)$ for each $x \in \tilde{X}$. Monotonicity implies that for any $x \geq y$, $\sup_{\tilde{X}} \tilde{B}(x) \vee \sup_{\tilde{X}} \tilde{B}(y) \in \tilde{B}(x)$. Therefore, $\tilde{b}^*(x) = \sup_{\tilde{X}} \tilde{B}(x) \geq \sup_{\tilde{X}} \tilde{B}(x) \vee \sup_{\tilde{X}} \tilde{B}(y) \geq \sup_{\tilde{X}} \tilde{B}(y) = \tilde{b}^*(y)$.

If $X = \tilde{X}$, noting that $X_{\geq x}$ is complete lattice for every x,[9] we can just use Theorem 5. As $X \neq \tilde{X}$, we will need an extra step:

Claim: Let x^* and $\tilde{x}^*$ be the largest fixed points of b^* and $\tilde{b}^*$. Then $\tilde{x}^* \geq x^*$.
Let $\tilde{X}_{\geq x^*} := \{x \in \tilde{X} \mid x \geq x^*\}$.
As $\tilde{X} \geq_{ss} X$, for any $x \in \tilde{X}$, $x^* \in X$, $x \vee x^* \in \tilde{X}$, and therefore $\tilde{X}_{\geq x^*}$ is nonempty. As $\tilde{X}$ is a complete lattice, so is $\tilde{X}_{\geq x^*}$.
Define $\tilde{g}^*$ on $\tilde{X}_{\geq x^*}$ as $\tilde{g}^*(x) = \tilde{b}^*(x)$. As $\forall x \in \tilde{X}_{\geq x^*}$, $x \geq x^*$, then $\tilde{g}^*(x) = \tilde{b}^*(x) \geq b^*(x^*)$, and

[8] This follows because the partial order on $X, \tilde{X}$ is the product order.
[9] The take any subset $S \subseteq X_{\geq x} \subseteq X$; we have $z := \inf_X S \in X$ (as X is a complete lattice). Noting that x is a lower bound for S and z is the greatest lower bound for S according to $\geq$, we have that $x \leq z \in X_{\geq x}$.

therefore $\tilde{g}^*$ is a self-map on a complete lattice.
As $\tilde{b}^*$ is monotone, so is $\tilde{g}^*$.
By Tarski's fixed point theorem, $\tilde{g}^*$ has a fixed point $y^* \in \tilde{X}_{\geq x^*}$, which must be also a fixed point of $\tilde{b}^*$, by definition of $\tilde{g}^*$. Then, $\tilde{x}^* = \sup_{\mathscr{F}(\tilde{B})} \mathscr{F}(\tilde{B}) \geq y^* \geq x^*$. □

Now let's go back to normal-form games $\Gamma = \langle I, X, u\rangle$, where I and X are as above and $u = (u_i)_{i\in I}$, with $u_i : X \to \mathbb{R}$ denoting player i's payoff. We define B_i as player i's best-response correspondence: $B_i(x_{-i}) := \arg\max_{x_i\in X_i} u_i(x_i, x_{-i})$.

Given two normal-form games, Γ and $\tilde{\Gamma}$, what do we need in order to guarantee that (i) $\tilde{B}_i, B_i$ are strong set monotone, (ii) $\tilde{B}_i(\tilde{x}_{-i})$ strong set dominates $B_i(x_{-i})$ for every $\tilde{x} \geq x$, and (iii) $\tilde{B}_i, B_i$ are complete-sublattice-valued? The answer to the first two points we already know from our lecture on monotone comparative statics of individual choices. Let's restate Milgrom and Shannon's (1994) Monoticity theorem:

Theorem 9. *(Monotonicity; (Milgrom and Shannon, 1994, Theorem 4)) Let X be a lattice and v, u be two real-valued functions on X. v and u are quasisupermodular and v single-crossing dominates u if and only if, for $S' \geq_{ss} S$, $X(S'; v) \geq_{ss} X(S; u)$.*

as well as the following corollary:

Corollary 2. *(Milgrom and Shannon, 1994, Corollary 2) Let X be a lattice, S a sublattice, and f a real-valued function on X. If f is quasisupermodular, then $X(S; f)$ is a sublattice of S.*

That is, it suffices that (a) $X_i, \tilde{X}_i$ to be (i) compact, and complete sublattices of a lattice Y_i, and (ii) $\tilde{X}_i \geq_{ss} X_i$; and (b) $u_i, \tilde{u}_i$ to be (i) quasisupermodular and continuous; and (ii) $\tilde{u}_i \geq_{sc} u_i$.

The existing literature focuses on changes in payoff functions. This gave rise to a very convenient class of games, with properties naturally closely related to the ones above.

Definition 5. A class of games $\{\Gamma(t)\}_{t\in T}$ has **strategic complementarities** if $\Gamma(t) = \langle I, X, u^t\rangle$, where I is finite, T is a partially ordered set, and, for all $i \in I$, X_i is a compact lattice, $u_i^t : X \to \mathbb{R}$ is upper semi-continuous in x_i and continuous in x_{-i}, quasisupermodular in x_i, and satisfies the single-crossing property in $(x_i; x_{-i}, t)$.

The games satisfying the assumptions above are usually known as *supermodular* games.

Exercise 4. *Consider an oligopolistic industry where each firm $i \in I$ simultaneously chooses a price $p_i \in [0, \overline{P}_i]$. Profits are given by $\pi_i(p_i, p_{-i}) := (p_i - c_i)D_i(p_i, p_{-i})$, where $D_i(p_i, p_{-i}) := a_i - bp_i + \sum_{j\neq i} d_{ij} p_j$.*

Show that the set of (pure strategy) Nash equilibria of the game is a nonempty complete lattice.

Games with strategic complementarities (also known as supermodular games) are particularly convenient since they provide an easy manner of characterising the largest and smallest Nash equilibria:

Proposition 1. *Let $\{\Gamma(t)\}_{t \in T}$ have strategic complementarities. For any t, let $X^{NE}(t)$ denote the set of pure Nash equilibria of $\Gamma(t)$.*
$X^{NE}(t)$ is a complete lattice, monotone wrt t in the strong set order.
Furthermore, for any t, the largest and and smallest Nash equilibria are the largest and smallest outcomes (resp.) survives IESDS.

Corollary 3. *(a) A supermodular game has a pure strategy Nash equilibrium*

(b) The greatest and least strategy profiles in the sets of (i) strategy profiles surviving IESDS, (ii) rationalisable strategy profiles, (iii) correlated equilibria, and (iv) Nash equilibria exist and are all the same.

(c) If a supermodular game has a unique Nash equilibrium, it is dominance solvable.

Exercise 5. *Prove Proposition 1 and Corollary 3.*

Note that Proposition 1 and Corollary 3 are not only of theoretical but also of practical interest. Specifically, these results allow you to obtain the greatest and smallest (pure strategy) equilibria via a simple iterative procedure. This renders computation of relevant Nash equilibria particularly simple.

Games with Weak Strategic Complementarities. We now provide weak set order counterparts to Theorems 7 and 8.

Theorem 10. *(Che et al., 2021, Theorem 9(i)) Let $X := \times_{i \in I} X_i$ be such that, $\forall i \in I$, X_i is a compact partially ordered metric space, and $B_i : X_{-i} \rightrightarrows X_i$ is nonempty- and compact-valued. If*

(i) $\exists x, y \in X$ such that, for every $i \in I$, $y_i \in B_i(x_{-i})$ and $y_i \geq x_i$ (resp. $\leq$), and

(ii) for every $i \in I$, B_i is upper (resp. lower) weak set monotone,

then the set of fixed points of B, $\mathscr{F}(B)$, is nonempty.

Proof. B_i is nonempty- and compact-valued by assumption. B is nonempty- and closed-valued, and upper (resp. lower) weak set monotone. Thus, by Theorem 3, $\mathscr{F}(B) \neq \emptyset$. □

Theorem 11. *(Che et al., 2021, Theorem 9(ii)) Let $X := \times_{i \in I} X_i$ be such that, $\forall i \in I$, X_i is a compact partially ordered metric space, and $B_i, \tilde{B}_i : X_{-i} \rightrightarrows X_i$ are nonempty- and compact-valued. If*

(i) $\exists x, y \in X$ such that, for every $i \in I$, $y_i \in \tilde{B}_i(x_{-i})$ and $y_i \geq x_i$ (resp. $\leq$),

(ii) for every $i \in I$, $\tilde{B}_i$ is upper (resp. lower) weak set monotone, and $\tilde{B}_i(x_{-i}) \geq_{uws} B_i(x_{-i})$ $\forall x_{-i} \in X_{-i}$ (resp. $\geq_{lws}$)

then $\mathscr{F}(\tilde{B}) \geq_{uws} \mathscr{F}(B)$ (resp. $\geq_{lws}$) whenever the latter is nonempty.

Proof. By Theorem 10, $\mathscr{F}(\tilde{B}) \neq \emptyset$. As $\tilde{B}$ upper (resp. lower) weak set dominates B, by Theorem 6, $\mathscr{F}(\tilde{B}) \geq_{uws} \mathscr{F}(B)$ (resp. $\geq_{lws}$). □

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Appendix A. Proof of Tarki's Fixed-Point Theorem (Theorem 1)

To show that $\mathscr{F}(f)$ has a smallest element, we need to take a small detour.

First, we need to introduce the concept of an **ordinal**, which generalises the cardinal numbers. Think about the, as an extension of the natural numbers. An ordinal is the concept that translates the size or cardinality of a set even when this set is not countable. For instance, $\mathbb{R}$, while uncountable, has a cardinality.

Definition 6. A binary relation $>$ on X is said to be **well-founded** if for every nonempty subset $S \subseteq X$, there is a **minimal element** y of S in S, that is, $y \in S$ and $\nexists x \in S$ such that $y > x$.

The ordinal numbers are not only well-founded set, but also totally ordered. (Again think about the natural numbers; are they well-founded? are they totally ordered?)

One important property of well-founded binary relations on sets is that we can apply a particular form of definition by recursion, called *transfinite recursion*. This goes as follows:

Theorem 12. *(Transfinite/Noetherian recursion) Let $(X, \geq)$ be a partially ordered set and $> \subseteq X \times X$ be well-founded. If, to each pair (x, g), where $x \in X$ and g is a function on $\{y \in X \mid y < x\}$, H assigns an object $H(x, g)$, then there is a unique function G on X such that, $\forall x \in X$, $G(x) = H(x, G|_{y \in X \mid y < x})$.*

Theorem 13. *(Transfinite/Noetherian induction) Let $(X, \geq)$ be a partially ordered set and $> \subseteq X \times X$ be well-founded. If $\forall x \in X$, property P being satisfied for all $y < x$ implies that P is satisfied for x, then P is satisfied for all $x \in X$.*

We know that we can apply recursion and induction on countable sets. Transfinite recursion an induction extend these notions to uncountable sets.

Lemma 5. *(Echenique, 2005, Lemma 1) Let X be a complete lattice and f be a self-map on X. If f is monotone, then $(\mathscr{F}(f), \geq)$ has a smallest element.*

Proof. Let η be an ordinal number with cardinality greater than X and define $\xi := \eta + 1$. Let $g : \xi \to X$ be defined by transfinite recursion as follows: $g(0) = \inf_X X$ and $g(\beta) = \sup_X \{f(g(\alpha)) \mid \beta > \alpha\}$, for $\beta > 0$. Now note that $f(g(0)) \geq g(0) (= \inf_X X)$, and therefore by transfinite induction $f(g(\beta)) \geq g(\beta)$ for all ordinals β, and we conclude that g is monotone. As ξ has cardinality larger than X, there is an ordinal $\gamma < \xi$ such that $g(\gamma) = g(\gamma + 1)$ which further implies that there is a smallest $\underline{\gamma}$ (the set of ordinals is well-founded and totally ordered) such that $f(g(\underline{\gamma})) = f(g(\underline{\gamma})) = g(\underline{\gamma})$. Therefore, the set of fixed points of $\tilde{f}$ is nonempty: $\mathscr{F}(f) \neq \emptyset$.

We now show that $g(\underline{\gamma})$ is the smallest fixed point of f. If $\underline{\gamma} = 0$ we are done, as $g(0) = \inf_X X$. If not, then $g(0) < f(g(0))$. Take any $z \in \mathscr{F}(f)$. As, by monotononity of f, $z = f(z) \geq f(g(0)) \geq g(0)$, then, by transfinite induction $z \geq f(g(\alpha))$ for any α. Hence, $z \geq f(g(\underline{\gamma})) = g(\underline{\gamma})$. □

Theorem 14. *(Tarski 1955; Echenique 2005, Theorem 2) Let X be a complete lattice and f be a self-map on X. If f is monotone, then $\mathscr{F}(f)$ is a non-empty complete lattice.*

Proof. By Lemma 5, $\mathscr{F}(f)$ is non-empty and has a smallest element. Let $S \subseteq \mathscr{F}(f)$ be non-empty. We want to show that $\sup_{\mathscr{F}(f)} S \in \mathscr{F}(f)$.

Let $y := \sup_X S$ and let $Y := \{x \in X \mid x \geq y\}$, be the set of upper bounds on S. If $y \in Y$, then $\forall x \in S$, $x \leq y \implies x = f(x) \leq f(y)$. Therefore, $f(Y) \subseteq Y$. Let $g = f\mid_Y$; g is then a monotone self-map on Y, a complete lattice. By Lemma 5, $(\mathscr{F}(g), \geq)$ has a smallest element, i.e. $\inf_X \mathscr{F}(g) \in \mathscr{F}(g) \subseteq \mathscr{F}(f) \cap Y$. By definition of g, $\min_X \mathscr{F}(g) = \sup_{\mathscr{F}(f)} S$. The proof for $\inf_{\mathscr{F}(f)} S$ is symmetric. □

Appendix B. Proof of Zhou's Fixed-Point Theorem (Theorem 2)

Lemma 6. *(Echenique, 2005, Lemma 3) Let X be a complete lattice and $F : X \rightrightarrows X$. If F is monotone and, $\forall x \in X$, $F(x)$ has a smallest element, then $(\mathscr{F}(F), \geq)$ has a smallest element.*

Proof. Let $f : X \to X$ be given by $f(x) := \min_X F(x)$. By Lemma 5, $(\mathscr{F}(f), \geq)$ has a smallest element; denote it z. By construction, $z \in \mathscr{F}(F)$, as $z = \min_X F(z)$.

To see that z is the smallest element in $(\mathscr{F}(F), \geq)$, take any $e \in \mathscr{F}(F)$. Let g, and $\underline{\gamma}$ be defined as in the proof for Lemma 5 with respect to f. Take any $e \in \mathscr{F}(F)$. The steps are then identical: As $g(0) \leq f(g(0)) \leq e \implies f(g(\alpha)) \leq e$ for every ordinal α, by transfinite induction $z = g(\underline{\gamma}) = f(g(\underline{\gamma})) \leq e$. □

Theorem 15. *(Zhou 1994, Theorem 1; Echenique 2005, Theorem 4) Let X be a complete lattice and $F : X \rightrightarrows X$. If F is monotone and, $\forall x \in X$, $F(x)$ is a complete sublattice, then $(\mathscr{F}(F), \geq)$ is nonempty complete lattice.*

Proof. Take any subset of fixed points $E \subseteq \mathscr{F}(F)$, which exist, by Lemma 6.
We now show that $\sup_{\mathscr{F}(F)} E$ exists. Let $x := \sup_{X)} E$ and $Y := \{y \in X \mid x \leq y\}$. Note that Y is a complete lattice, as $\forall S \subseteq X$, $x \leq \sup_X S \in X \implies \sup_X S \in Y \implies \sup_X S = \sup_Y S$; an analogous argument holds for $\inf_Y S$.

Define $G : Y \rightrightarrows Y$ as $G(y) := Y \cap F(y)$; this is the set of elements in $F(y)$ that are weakly larger than x. The goal will be to show that G is (i) nonempty, (ii) monotone, and (iii) a complete sublattice of Y. Then we (iv) apply the previous lemma and show that it has a smallest point in Y, and (v) conclude by saying that this fixed is also a fixed point in X and that it must be in E. We now prove each of these claims:

(i) We want to show that $G(y) \neq \emptyset$. $e \leq x \leq y$, $\forall e \in E$ and $y \in Y$. Then, $\forall x_y \in F(y)$ and as $e \in F(e)$, $x_y \vee e \in F(y)$ by monotonicity of F. Fix an arbitrary $x_y \in F(y)$ and note that $\sup_X S =: x \leq \sup_X \{x_y \vee e \in X \mid e \in E\} \in F(y)$ as F is a complete sublatttice of X (by assumption). Hence, $G(y) := Y \cap F(y) \neq \emptyset$.

(ii) Now we show that G is monotone. Take any $y' \leq y \in Y$, $z \in G(y)$ and $z' \in G(y')$. Note that $y \wedge y' \in F(y)$ and $y \vee y' \in F(y')$ as F is monotone. Furthermore, $\forall e \in E$, $e \in F(e)$, $e \leq y \leq y'$, and thus, $y \wedge y' = (y \wedge y') \vee e \in F(y)$ and $y \vee y' = (y \vee y') \vee e \in F(y')$, again by monotonicity of F.

(iii) $\forall S \subseteq G(y)$, $\inf_Y S, \sup_Y S \in G(y)$. To see this note that as F is a complete sublattice, and as $S \subseteq G(y) = F(y) \cap Y$, $S \subseteq F(y)$, and then $\sup_Y S = \sup_X S \cap Y \in F(y)$, and analogously for $\inf_Y S$. As $\forall z \in S \subseteq Y$, $z \geq x$, $\sup_X S \cap Y \in Y \implies \sup_X S \cap Y \in F(y) \cap Y = G(y)$; and analogously for the $\inf_Y \in S$.

(iv) All the hypotheses of Lemma 6 are satisfied, and therefore let $y^* \in \mathscr{F}(G)$ be the smallest fixed point of G in Y. For any fixed point x^* of F in X that is an upper bound on E, we must have that $x \leq x^* \implies x^* \in F(x^*) \cap Y = G(x^*)$, and therefore it is a fixed point of G in Y. Hence, $x^* \geq y^*$. As $y^* \in G(y^*) = F(y^*) \cap Y$, y^* is also a fixed point of F in X. We conclude that $y^* = \sup_{\mathscr{F}(F)} E$.

The proof that $\sup_{\mathscr{F}(F)} E$ exists is symmetric. □

Appendix C. Proof Sketch of LCKK's Fixed-Point Theorem (Theorem 3)

A very rough sketch of the main steps in the proof is as follows:

(1) X being a compact metric space implies that each nonempty chain has a least upper bound.

(2) Upper weak set monotonicity of F together with compactness allows for a proof for existence of a fixed point with the same spirit to finding the largest fixed point in Lemma 2 by doing a monotone selection of F.

(3) For existence of a maximal fixed point, take any chain of fixed points. Compactness implies that it admits a smallest upper bound and closed-valuedness and uws-monotocity of F will (eventually — the proof is complex) yield that the smallest upper bound is a fixed point.

(4) As (3) holds for any chain of fixed points, appeal to Zorn's Lemma to claim that there is a maximal fixed point.