ECON0106: **Microeconomics**

2. Structural Properties of Preferences and Utility Representations[*]

Duarte Gonçalves[†]

University College London

1. Overview

In many circumstances, economic research involves taking a stance on the form of utility functions that drive agents' behaviour. This is true in applied research — where models are then used to analyse policies with structural estimation and counterfactuals, or to identify a particular effect — as it is in theory. Often, making simplifying assumptions are needed to make progress, but it is important to keep in mind what each assumption implies, and what it is ruling out. Our goal will be to develop a better understanding about common restrictions that specific functional form assumptions impose on behaviour to be able to better evaluate the limitations of any given model.

1.1. Notation

For simplicity, assume throughout that (X,d) is a metric space. For $\epsilon > 0$ and $x \in X$, we denote by $B_\epsilon(x) := \{y \in X \mid d(x,y) < \epsilon\}$ for an open ϵ-neighbourhood of x in X. For a set S, we denote its closure by $\overline{S}$.

2. Continuous Utility Representation

Why do we care about continuity[1] of our utility representation? Because, whenever the feasible set is compact,[2] we are guaranteed the existence of a maximiser by Weierstrass extreme

[*]Last updated: 23 September 2025.

[†] Department of Economics, University College London; duarte.goncalves@ucl.ac.uk. Please do not share these notes with people outside of this class.

[1]Recall the definition: Let (X,d_X) and (Y,d_Y) be two metric spaces. A function $f: X \to Y$ is **continuous** at $x_0 \in X$ if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $\forall x \in \{x' \in X : d_X(x', x_0) < \delta\}$, $d_Y(f(x), f(x_0)) < \epsilon$. Equivalently, f is continuous at $x_0 \in X$ if for every sequence $\{x_n\}_n \subseteq X$ such that $x_n \to x_0$, $f(x_n) \to f(x_0)$. A function is continuous if it is continuous at all $x_0 \in X$.

[2]When (X,d_X) is a metric space, a set S is **compact** if and only if it is **sequentially compact**, that is, if every sequence $\{x_n\}_n$ has a convergent subsequence $\{x_m\}_m \subseteq \{x_n\}_n$ such that $x_m \to x^* \in S$. Note that if $S \subseteq Y \subseteq X$, S is closed, and Y is compact, then S is compact. Moreover, if S is a compact set, then S is both closed and bounded. The **Heine–Borel Theorem** shows that, in Euclidean spaces, the converse holds: if S is closed and bounded, then S is compact.

value theorem:

Theorem 1. ***(Weierstrass Extreme Value Theorem)*** *Let (X, d_X) and (Y, d_Y) be two metric spaces. If $f : X \to Y$ is a continuous function and S a compact set in (X, d_X), then f attains a maximum and a minimum in S: $\arg\max_{x \in S} f(x) \neq \emptyset$ and $\arg\min_{x \in S} f(x) \neq \emptyset$.*

We want to guarantee that utility maximising choices are well-defined ($\arg\max_{x \in A} u(x) \neq \emptyset$) in order to ensure that an agent's behaviour can be represented as the outcome of utility maximisation. And, in general, one way to do so is by restricting to compact feasible sets and continuous utility functions.

In this section we will relate continuity properties of the utility representation to underlying continuity properties of preferences and choices.

Definition 1. A preference relation $\succsim$ on X is **continuous** if for any two converging sequences, $\{x_n\}_n, \{y_n\}_n \subseteq X$, $x_n \to x$ and $y_n \to y$, such that $x_n \succsim y_n$ $\forall n$, we have $x \succsim y$.

This next lemma is particularly useful to characterise continuity of preference relations:

Lemma 1. *Let $\succsim$ be a preference relation on X, and $\succ$ its asymmetric part. The following statements are equivalent:*

(i) $\succsim$ is continuous;

(ii) for any $x \in X$, $X_{x\succsim}$ and $X_{\succsim x}$ are closed sets;

(iii) for any $x \in X$, $X_{x\succ}$ and $X_{\succ x}$ are open sets;

(iv) for any $x, y \in X$ such that $x \succ y$, there is $\epsilon > 0$ such that $\forall x' \in B_\epsilon(x), y' \in B_\epsilon(y)$, $x' \succ y'$.

Exercise 1. *Prove Lemma 1. (Hint: a standard way to go about it is to show (i) $\implies$ (ii) $\implies$ (iii) $\implies$ (iv) $\implies$ (i).)*[3]

The main result in this section is Debreu's (1954; 1964) Theorem:

Theorem 2. ***(Debreu's Theorem)*** *Let $\succsim$ be a preference relation on X, and suppose that X admits a countable, order-dense subset Z. Then, $\succsim$ is continuous if and only if $\succsim$ admits a continuous utility representation $u : X \to \mathbb{R}$.*

We won't prove the theorem in all its generality, but rather a more modest version of it in which we will assume that X is convex, that is, $\forall x, y \in X$ and $\forall \lambda \in [0,1]$, $\lambda x + (1-\lambda) y \in X$:

Theorem 3. *Let (X, d) be a convex metric space such that $\forall \alpha \in [0,1]$, $\forall x, y \in X$, $d(\alpha x + (1-\alpha) y, y) \leq \alpha d(x,y)$. Let $\succsim$ be a preference relation on a convex set X, and suppose that X admits a countable, order-dense subset Z. Then, $\succsim$ is continuous if and only if $\succsim$ admits a continuous*

[3] You can think about $\implies$, i.e., "implies", as a preorder on the set of all statements, which is why if $A \implies B \implies C \implies A$, we have, by transitivity, $A \implies C$, $C \implies B$, and $B \implies A$, and, hence, that A, B, and C are equivalent.

utility representation $u : X \to \mathbb{R}$.

Proof. $\Longleftarrow$: (if)
Take any $\{x_n\}_n, \{y_n\}_n \subseteq X$ such that $x_n \to x$, $y_n \to y$, and $x_n \succsim y_n$. Then, $u(x_n) - u(y_n) \geq 0$, $\forall n$ and, by continuity of u, $\lim_{n\to\infty} u(x_n) - u(y_n) = u(x) - u(y) \geq 0 \implies x \succsim y$.

$\Longrightarrow$: (only if)
Assume that $\exists x, y \in X : x \succ y$ — otherwise we can just set $u(x) = c$ for any constant c.
We will prove this part in three steps:

1. Show that $\forall x, y \in X : x \succ y$, there is $z \in Z$ such that $x \succ z \succ y$.
2. Construct a utility function $u : X \to \mathbb{R}$ such that $u(Z)$ is dense in $[0,1]$.
3. Show that u is continuous.

Lemma 2. *Let (X,d) be a convex metric space such that $\forall \alpha \in [0,1]$, $\forall x, y \in X$, $d(\alpha x + (1-\alpha)y, y) \leq \alpha d(x,y)$. Let $\succsim$ be a continuous preference relation on X, and $\succ$ its asymmetric part. Suppose that Z is a countable, order-dense subset of X. For any $x, y \in X : x \succ y$, there is $z \in Z$ such that $x \succ z \succ y$.*

Proof. First we show existence of $x' \in X : x \succ x' \succ y$.
For $a \in [0,1]$, let $x_a := ax + (1-a)y \in X$ (by convexity of X). Define $A := \{a \in [0,1] \mid x_a \succsim x\}$.
By completeness of $\succsim$, $1 \in A$ and, by assumption, as $x \succ y$, $0 \notin A$. Then, as A is nonempty and bounded below, define $\alpha := \inf A$.
We now want to show that $x_\alpha \sim x$. Suppose that this is not the case.

$$\begin{aligned} \text{If } x_\alpha \succ x &\implies \exists \epsilon > 0 : x_{\alpha-\epsilon} \succ x && \text{by Lemma 1} \\ &\implies \alpha - \epsilon \in A \\ &\implies \alpha \neq \inf A, \end{aligned}$$

a contradiction. If instead,

$$\begin{aligned} x \succ x_\alpha &\implies \exists \epsilon' > 0 : x \succ x_{\alpha+\epsilon},\ \forall 0 < \epsilon \leq \epsilon' && \text{by Lemma 1} \\ &\implies \alpha + \epsilon > \inf A \\ &\implies \alpha \neq \inf A, \end{aligned}$$

again a contradiction. As $x \sim x_\alpha \succ y$, this means that, again, $\exists \epsilon > 0 : x_{\alpha-\epsilon} \succ y$. As $\alpha - \epsilon < \alpha \implies \alpha - \epsilon \notin A$, and, therefore, $x \succ x_{\alpha-\epsilon} \succ y$.

Now, we find a $z \in Z$ such that $x \succ z \succ y$: as Z is order-dense in X, then $\exists z \in Z : x \succ x_{\alpha-\epsilon} \succsim z \succ y$. □

Lemma 3. *Let (X,d) be a convex metric space such that $\forall \alpha \in [0,1]$, $\forall x, y \in X$, $d(\alpha x + (1-\alpha)y, y) \leq \alpha d(x,y)$. Let $\succsim$ be a continuous preference relation on X, and Z a countable order-*

dense subset of X. There is a utility representation of $\succsim$ $u : X \to \mathbb{R}$ such that $u(X)$ is dense in $[0,1]$.

Proof. Without loss of generality, suppose that the maximal and minimal elements of X are not in Z, i.e., $Z \cap (\arg\max_{\succsim} X \cup \arg\min_{\succsim} X) = \emptyset$. Fix an order on $Z = \{z_1, z_2, \ldots\}$ and, for $n \geq 2$, let $Z_n := \{z_1, \ldots, z_{n-1}\}$. We define u on Z by induction. Let $u(z_1) = 1/2$. For any $n > 1$, (i) if $\exists z_m \in Z_n$ such that $z_n \sim z_m$, set $u(z_n) = u(z_m)$; (ii) if $z_n \succ z_m$ (resp. $z_m \succ z_n$) for all $z_m \in Z_n$, then set $u(z_n) := (1 + \max_{z \in Z_n} u(z))/2$ (resp. $u(z_n) := (0 + \min_{z \in Z_n} u(z))/2$); and (iii) if neither (i) nor (ii) hold, then $\exists z_\ell, z_m \in Z_n$ such that (a) $z_\ell \succ z_n \succ z_m$ and (b) $\nexists z' \in Z_n : z_\ell \succ z \succ z_n$ nor $z_n \succ z \succ z_m$, in which case set $u(z_n) := (\min_{z \in Z_n : z \succ z_n} u(z) + \max_{z \in Z_n : z_n \succ z} u(z))/2$. Note that, by Lemma 2, $\forall x, y \in X : x \succ y$, $\exists z \in Z : x \succ z \succ y$. This implies that for any two elements $z_n, z_m \in Z$ such that $z_n \succ z_m$, there is $\ell, \ell', \ell'' > n, m$ such that $z_\ell \succ z_n \succ z_{\ell'} \succ z_m \succ z_{\ell''}$, where z_ℓ and $z_{\ell''}$ exist because we removed the maximal and minimal elements of X from Z.
By construction, the set $u(Z)$ corresponds to the set of dyadic numbers in $(0,1)$, i.e., the set of numbers that can be represented as $m/2^n$ for $m, n \in \mathbb{N}$ and $m < 2^n$, which is dense in $[0,1]$.
If $\arg\max_{\succsim} X$ is nonempty, we know that $\forall x, y \in \arg\max_{\succsim} X$, we have that $x \sim y$, and then we can assign $u(x) = u(y) = 1$; and analogously for $x \in \arg\min_{\succsim} X$ assign $u(x) = 0$.

Now we extend u to X by setting $u(x) := \sup\{u(z) \mid z \in Z \text{ and } x \succ z\} = \sup_{z \in Z_{x \succ}} u(z)$ and check that it represents $\succsim$. That $x \sim y \implies u(x) = u(y)$ is immediate from the definition. To see that $x \succ y \implies u(x) > u(y)$, note that, by Lemma 2, $\exists z, z' \in Z$ such that $x \succ z \succ z' \succ y \implies u(x) \geq u(z) > u(z') \geq u(y)$. As $u(Z) \subseteq u(X) \subseteq [0,1]$ and $u(Z)$ is dense in $[0,1]$, then $u(X)$ is dense in $[0,1]$. □

Finally, the last step: showing continuity of u as defined in the proof of Lemma 3. Take any $x \in X \setminus (\arg\max_{\succsim} X \cup \arg\min_{\succsim} X)$. By 2Lemmas and 3, for any $\epsilon > 0$, there are $z, z' \in Z$ such that $u(x) - \epsilon < u(z) < u(x) < u(z') < u(x) + \epsilon$. By Lemma 1, we then have that $\exists \delta > 0$ such that $\forall x' \in B_\delta(x)$, $u(x) - \epsilon < u(z) < u(x') < u(z') < u(x) + \epsilon$. To show that u is continuous at $x \in \arg\max_{\succsim} X$, note that by 2Lemmas and 3, for any $\epsilon > 0$, there is $z \in Z$ such that $u(x) - \epsilon < u(z) < u(x)$ and by Lemma 1, $\exists \delta > 0$ such that $\forall x' \in B_\delta(x)$, $u(x) - \epsilon < u(z) < u(x') \leq u(x)$ (the proof for continuity of u at $x \in \arg\min_{\succsim} X$ is symmetric). □

Exercise 2. *Prove that, if X is a convex subset of $\mathbb{R}^k$ and $\succsim$ is a continuous preference relation on X, then X admits a countable, order-dense subset. Conclude about the existence of a continuous utility representation.*

Note that even if $\succsim$ is a continuous preference relation on X, it does not mean that any utility representation of $\succsim$ is continuous. For instance, suppose that $\succsim$ is a continuous preference relation on $\mathbb{R}$ such that $x \geq y \iff x \succsim y$. Clearly, $u(x) := x$ is a possible utility representation, but so is $v(x) := x$ if $x < 1$, $v(x) := 3x$ if $x > 1$, and $v(1) := 2$, and v is not continuous.

3. Convexity

If the former section dealt with having the agent's choices well-defined for arbitrary compact sets, this section will provide sufficient conditions for choices to be uniquely defined (i.e., a singleton). For that, we will study a property of interest for a utility representation — quasi-concavity — and the conditions on preferences that guarantee it.

Definition 2. A real-valued function u on a convex set X is **(strictly) quasiconcave** if $\forall x, y \in X$ and $\forall \lambda \in [0,1]$ (resp. $\lambda \in (0,1)$), $u(\lambda x + (1-\lambda y)) \geq (>) \min\{u(x), u(y)\}$.

Definition 3. We say that a preference relation $\succsim$ on a convex set X is **convex** iff for any $x \succsim y$ and any $\lambda \in [0,1]$, we have that $\lambda x + (1-\lambda)y \succsim y$. It is said to be **strictly convex** if, in addition, $\forall x \succsim y$, $x \neq y$, and any $\lambda \in (0,1)$, $\lambda x + (1-\lambda)y \succ y$.

This property can be interpreted as prefering mixtures to extremes (when indifferent with respect to both): when the agent is indifferent between two elements (say "apple" and "banana") then they prefer to have a convex combination (λ"apple"+$(1-\lambda)$"banana"=:"fruit salad") to either of them.

In fact, quasiconcavity of the utility representation is equivalent to convexity of preferences:

Proposition 1. *Let $\succsim$ be a preference relation on a convex set X and let $u : X \to \mathbb{R}$ be a utility representation. The following statements are equivalent:*

(i) $\succsim$ is convex;

(ii) $X_{\succsim y}$ is convex $\forall y \in X$;

(iii) u is quasiconcave;

(iv) $\{x \in X \mid u(x) \geq \overline{u}\}$ is convex $\forall \overline{u} \in \mathbb{R}$.

Moreover, $\succsim$ is strictly convex if and only if u is strictly quasiconcave.

Proof. (i) $\Longrightarrow$ (ii): Take any $x, x' \in X_{\succsim y}$ and let, without loss of generality (by completeness), $x \succsim x'$. Then $\lambda x + (1-\lambda)x' \succsim x' \succsim y$ $\forall \lambda \in [0,1]$ (by convexity and transitivity).
(i) $\Longleftarrow$ (ii): By completeness, $y \in X_{\succsim y}$. As $X_{\succsim y}$ is convex, then $\forall x \in X_{\succsim y}$ and $\lambda \in [0,1]$, $\lambda x + (1-\lambda)y \succsim y$.
(i) $\Longleftrightarrow$ (iii): Take any $x, y \in X$ such that $x \succsim y \iff u(x) \geq u(y)$, and any $\lambda \in [0,1]$.

$$\begin{aligned} \succsim \text{ convex } &\Longleftrightarrow \lambda x + (1-\lambda)y \succsim y \\ &\Longleftrightarrow u(\lambda x + (1-\lambda)y) \geq u(y) = \min\{u(x), u(y)\} \\ &\Longleftrightarrow u \text{ quasiconcave.} \end{aligned}$$

For the strict convexity of $\succsim$ and strict quasiconcavity of u, replace $\succsim$ and $\geq$ with $\succ$ and $>$.
(iii) $\Longrightarrow$ (iv): $\forall x, y \in X : u(x), u(y) \geq \overline{u}$, $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\} \geq \overline{u}$, $\forall \lambda \in [0,1]$ (by

quasiconcavity of u).

(iii) $\Longleftarrow$ (iv): $\forall x,y \in X$, $\lambda \in [0,1]$, $\lambda x + (1-\lambda)y \in \{z \in X \mid u(z) \geq \min\{u(x),u(y)\}\}$ by convexity of $\{z \in X \mid u(z) \geq \min\{u(x),u(y)\}\}$ and the fact that $u(x),u(y) \geq \min\{u(x),u(y)\}$; then $u(\lambda x + (1-\lambda)y) \geq \min\{u(x),u(y)\}$. □

Theorem 4. *Let $\succsim$ be a convex preference relation on a convex set X. Then, for any convex $A \in 2^X$, $\arg\max_{\succsim} A$ is convex. If, in addition, $\succsim$ is strictly convex, then $\arg\max_{\succsim} A$ contains at most one element.*

Combining Proposition 1 and Theorem 4, we learn that the set of maximisers of a quasiconcave function u on a convex set A, $\arg\max_{x \in A} u(x)$, is convex and, if u is strictly quasiconcave, it has at most one element. This gives us a lot of structure on the agent's choices and allows us to make better predictions. In fact, pooling results we have shown so far, we can say that if preferences are continuous and strictly convex, the agent will choose exactly one element out of any compact and convex set of alternatives.

Exercise 3. *Prove Theorem 4.*

4. Monotonicity and Insatiability

A very natural property when $X \subseteq \mathbb{R}^k$ is that of monotonicity, capturing the principle that 'more is better.' We define three notions of monotonicity:

Definition 4. (i) $\succsim$ is **monotone** if $x \geq y \implies x \succsim y$;

(ii) $\succsim$ is **strongly monotone** if $x \geq (\gg) y \implies x \succsim (\succ) y$;[4]

(iii) $\succsim$ is **strictly monotone** if $x > y$ (i.e., $x \geq y$ and $x \neq y$) $\implies x \succ y$.

A simple result ensues:

Proposition 2. *Let $\succsim$ be a preference relation on $X \subseteq \mathbb{R}^k$ and $u : X \to \mathbb{R}$ a utility representation of $\succsim$.*

*(i) $\succsim$ is **monotone** if and only if $x \geq y \implies u(x) \geq u(y)$;*

*(ii) $\succsim$ is **strongly monotone** if and only if $x \geq (\gg) y \implies u(x) \geq (>) u(y)$;*

*(iii) $\succsim$ is **strictly monotone** if and only if $x > y$ ($x \geq y$ and $x \neq y$) $\implies u(x) > u(y)$.*

A related property is that of insatiability, the sense in which, for any alternative x, there is always some other alternative y that is strictly preferred.

Definition 5. (i) We say that $\succsim$ is **globally non-satiated** if for any $x \in X$, there is $y \in X$ such

[4] It is to be understood that $x \gg y$ stands for $x_i > y_i$ for all $i \in [k]$, whereas $x > y$ simply denotes the asymmetric part of $\geq$, $x \geq y$ and $\neg(y \geq x)$. That is, $x > y$ if $x_i \geq y_i$ for all i and $x_j > y_j$ for some j.

that $y \succ x$.

(ii) It is **locally non-satiated** if for any $x \in X$, and any $\epsilon > 0$, $\exists y \in B_\epsilon(x)$ such that $y \succ x$.

Then we have that strict monotonicity $\implies$ strong monotonicity $\implies$ monotonicity and, if, say, $X = \mathbb{R}^k$, strong monotonicity $\implies$ local non-satiation $\implies$ global non-satiation.[5] While non-satiation does not easily translate into properties of utility representations, we will see later on that it plays an important role in consumer theory.

5. Homotheticity

Definition 6. A preference relation $\succsim$ on $X = \mathbb{R}^k$ is **homothetic** if $x \succsim y \implies \alpha x \succsim \alpha y$, $\forall \alpha \geq 0$.

Homotheticity of preferences will allow us to show that one can interpret aggregate demand as choices by a representative consumer. We will defer on that result and instead focus on its implications for utility representation.

Proposition 3. *Let $\succsim$ be a continuous, homothetic, and strongly monotone preference relation on $X = \mathbb{R}^k$. Then, it admits a continuous utility representation $u : X \to \mathbb{R}$ that is homogeneous of degree one.*[6]

Exercise 4. *Prove Proposition 3 by following the following steps:*

(1) Show that, for any $x \in X$ there is an $\alpha, \alpha' \in \mathbb{R}$ such that $\alpha \mathbf{1} \succsim x \succsim \alpha' \mathbf{1}$, where $\mathbf{1} \in \mathbb{R}^k$ is a vector of ones.

(2) Show then that, for any $x \in X$, there is a unique $\beta_x \in \mathbb{R}$ such that $x \sim \beta_x \mathbf{1}$.

(3) Define $u(x) := \beta_x$ and show that u is continuous and homogeneous of degree one.

6. Separability

It is often the case that alternatives have different features. For instance, consider subscribing to a gym, and you consider the location and the open hours. It may be the case that you consider these two features separately, and then simply trade-off between the two. Or it may be the case that if the gym is open and not too crowded after work hours, you prefer it to be close to work as you can go with friends from work; but if it is usually too crowded, you actually prefer it to be close to home. When preferences are separable, the problem in a sense becomes simply a matter of assigning a value to each dimension and think about how you trade these off. This is exactly what we are going to show.

[5] In general, monotonicity need not imply non-satiation: consider the trivial case where X is a singleton.

[6] That is, $u(\alpha x) = \alpha u(x)$, $\forall \alpha \geq 0$.

Let $X := \times_{i\in[n]} X_i \times \overline{X}$, where each X_i is a dimension and $[n] = \{1,...,n\}$. We write $x_{-i} \in X_{-i} := \times_{j\in[n]\setminus\{i\}} X_j \times \overline{X}$ and $x = (x_i, x_{-i})$.

Definition 7. A preference relation on X is said to be **weakly separable** in $\times_{i\in[n]} X_i$ if, $\forall i \in [n]$, for every $x_i, y_i \in X_i$ and every $x_{-i}, y_{-i} \in X_{-i}$, we have that $(x_i, x_{-i}) \succsim (y_i, x_{-i}) \iff (x_i, y_{-i}) \succsim (y_i, y_{-i})$.

Theorem 5. *Let $\succsim$ be a preference relation on $X = \times_{i\in[n]} X_i \times \overline{X}$ that admits a utility representation $u : X \to \mathbb{R}$. Then, $\succsim$ is weakly separable in $\times_{i\in[n]} X_i$ if and only if there are $v, \{u_i\}_{i\in[n]}$, where $u_i : X_i \to \mathbb{R}$, and $v : \times_{i\in[n]} u_i(X_i) \times \overline{X} \to \mathbb{R}$ such that $u(x) = v(u_1(x_1),...,u_n(x_n),\overline{x})$ and v is strictly increasing in its first n arguments.*

Proof. $\Longleftarrow$: (if) Follows immediately from the fact that v is strictly increasing in its first n arguments.
$\Longrightarrow$: (only if) We break the proof into steps:

(1) Define u_i: Fix $x^* \in X$. For $i \in [n]$, let $u_i(x_i) := u(x_i, x^*_{-i})$.

(2) Show that, for any $x, y \in X$ such that $\overline{x} = \overline{y}$, if $u_i(x_i) \geq u_i(y_i)$ $\forall i \in [n]$, then $u(x) \geq u(y)$:

$$\begin{aligned}
u_i(x_i) \geq u_i(y_i)\,\forall i \in [n] &\iff u(x_i, x^*_{-i}) \geq u(y_i, x^*_{-i})\,\forall i \in [n] \\
&\iff (y_1,...,y_{i-1},x_i,x_{i+1},...,x_n,\overline{x}) \succsim (y_1,...,y_{i-1},y_i,x_{i+1},...,x_n,\overline{x})\,\forall i \in [n] \\
&\qquad\text{by weak separability} \\
&\implies x \succsim y \\
&\qquad\text{by transitivity} \\
&\implies u(x) \geq u(y).
\end{aligned}$$

Moreover, it is also the case that if for some i $u_i(x_i) > u_i(y_i)$, then $u(x) > u(y)$.

(3) Define v: For any $r \in \mathbb{R}^n$ such that $r_i \in u_i(X_i)$ $\forall i \in [n]$, pick any $x_i \in X_i$ such that $u_i(x_i) = r_i$. For any $\overline{x} \in \overline{X}$, and for any $r \in \times_{i\in[n]} u_i(X_i)$, let $v(r, \overline{x}) := u(x)$. By (2), v is strictly increasing in r.

□

Note that weak separability *does not* deliver additive separability, that is, it does not guarantee that we can write $u(x) = \sum_{i\in[n]} u_i(x_i)$. For that we need preferences to be **strongly separable** on $X = \times_{i\in[n]} X_i$:

Definition 8. A preference relation $\succsim$ on $X = \times_{i\in[n]} X_i$ is **strongly separable** if $\forall I \subsetneq [n]$, $\forall x_I, y_I \in \times_{i\in I} X_I$ and $\forall x_{-I}, y_{-I} \in \times_{i\in[n]\setminus I} X_i =: X_{-I}$, we have that $(x_I, x_{-I}) \succsim (y_I, x_{-I}) \iff (x_I, y_{-I}) \succsim (y_I, y_{-I})$.

In essence, strongly separable preferences are those that are separable not only in each dimension but in each group of dimensions.

We will also need a further definition: $i \in [n]$ is an **essential component** if $\exists x_i, y_i \in X_i$ and $x_{-i} \in X_{-i}$ such that $(x_i, x_{-i}) \succ (y_i, x_{-i})$. The result — which we state without proof — is then as follows:

Theorem 6. ***(Debreu, 1960)*** *Let $\succsim$ be a continuous preference relation on a connected set $X := \times_{i\in[n]} X_i$, such that $\succsim$ admits a preference relation $u : X \to \mathbb{R}$, and there are at least three essential components. If $\succsim$ is strongly separable, then there are $\{u_i\}_{i\in[n]}$, where $u_i : X_i \to \mathbb{R}$, such that $u(x) = \sum_{i\in[n]} u_i(x_i)$.*

7. Quasilinearity

One of the most widely used functional forms is **quasilinear utility**: $u : Y \times \mathbb{R} \to \mathbb{R}$ such that $u(y, m) = v(y) + m$ for some $v : Y \to \mathbb{R}$. The first argument y is interpreted as an item, while the second argument m is taken to be money (available to acquire other items). As it is a recurrently assumed functional form — e.g., in contract theory, auctions, and mechanism design — it is particularly important to understand precisely what assumptions on preference relations are necessary to obtain this type of representation.

Theorem 7. *Let $\succsim$ be a preference relation on $Y \times \mathbb{R}$. $\succsim$ admits a quasilinear utility representation if and only if it satisfies the following properties:*

(1) $m' \geq m \iff (y, m') \succsim (y, m)$, $\forall y \in Y$, $m, m' \in \mathbb{R}$ (money is good);

(2) $(y, m) \succsim (y', m') \iff (y, m + m'') \succsim (y', m' + m'')$, $\forall y, y' \in Y$, $m, m', m'' \in \mathbb{R}$ (no wealth effects);

(3) $\forall y, y' \in Y$, $\exists m, m' \in \mathbb{R}$ such that $(y, m) \sim (y', m')$ (money can compensate).

Note that property (2) is needed in some way or another: after all, quasilinear preferences are weakly separable.

Proof. $\implies$ (only if):

(1) $m' \geq m \iff v(y) + m' \geq v(y) + m \iff (y, m') \succsim (y, m)$, $\forall y \in Y$, $m, m' \in \mathbb{R}$ (money is good);

(2) $(y, m) \succsim (y', m) \iff v(y) + m \geq v(y') + m \iff v(y) + m' \geq v(y') + m' \iff (y, m') \succsim (y', m')$, $\forall y, y' \in Y$, $m, m' \in \mathbb{R}$ (no wealth effects);

(3) $\forall y, y' \in Y$, $\exists m, m' \in \mathbb{R}$ such that $v(y) - v(y') = m' - m \iff v(y) + m = v(y') + m' \iff (y, m) \sim (y', m')$ (money can compensate).

$\impliedby$ (if):

Fix $(y^*, m^*) \in Y \times \mathbb{R}$.

Step 1: there is a unique $\rho : Y \to \mathbb{R}$ such that $(y, \rho(y)) \sim (y^*, m^*)$.

By (3), $\exists m(y), m'(y) \in \mathbb{R} : (y, m(y)) \sim (y^*, m'(y))$. By (2), $(y, m(y) - m'(y) + m^*) \sim (y^*, m^*)$. Let $\rho(y) := m(y) - m'(y) + m^*$. Suppose there is $\nu : Y \to \mathbb{R}$ such that $(y, \nu(y)) \sim (y^*, m^*)\ \forall y \in Y$, but $\nu(y') \neq \rho(y')$ for some $y' \in Y$. Then $\nu(y') < (>) \rho(y')$ implies by (1) that $(y', \nu(y')) \sim (y^*, m^*) \sim (y', \rho(y')) \succ (\prec)(y', \nu(y'))$, a contradiction of reflexivity of $\succsim$.

Step 2: Show that we can define $v(y) := -\rho(y)$.

$$\begin{aligned}
(y, m) \succsim (y', m') &\iff (y, m - m' + \rho(y')) \succsim (y', \rho(y')) \sim (y^*, m^*) && \text{by Step 1 and (2)} \\
&\iff m - m' + \rho(y') \geq \rho(y) && \text{by (1)} \\
&\iff -\rho(y) + m \geq -\rho(y') + m' \\
&\iff v(y) + m \geq v(y') + m'.
\end{aligned}$$

□

8. Indifference Curves

What are indifference curves? At their very core, indifference curves are indifference sets: $[x]_\sim := \{y \in X \mid y \sim x\}$. If $X = \mathbb{R}^2_+$, let's define the 'indifference curve' that goes through y as a function $I(\cdot, y) : \mathbb{R} \to \mathbb{R}$ such that $I(x_1, y) := \{x_2 \in \mathbb{R} \mid (x_1, x_2) \in [y]_\sim\}$.

Exercise 5. *Suppose that $X = \mathbb{R}^2_+$. Which properties on $\succsim$ guarantee that indifference curves satisfy each of the following:*

(i) have an empty interiour;

(ii) are continuous;

(iii) are downward sloping;

(iv) are convex.

Exercise 6. *Let $\succsim$ be a preference relation on X. Suppose that theorem A states that if $\succsim$ satisfies property P_A, then there is a utility representation that satisfies property Q_A and that, similarly, theorem B states that if $\succsim$ satisfies property P_B, then there is a utility representation that satisfies property Q_B. Now suppose that you find that $\succsim$ satisfies both P_A and P_B. Can we guarantee that there is a utility representation satisfying both Q_A and Q_B?*

9. Further Reading

Standard References: Mas-Colell et al. (1995, Chapter 3C), Rubinstein (2018, Chapters 2, 4), Kreps (2012, Chapter 2), Kreps (1988, Chapter 3).

Related questions/topics: Besides the topics mentioned on the previous notes, structural properties of preferences are closely related to applications. In particular, with deriving objects such as price indices, aggregation of preferences and demand systems, as well as with growth and structural transformation.

References

Debreu, Gérard. 1954. "Representation of a Preference Ordering by a Numerical Function." In *Decision Processes*, edited by Thrall, M., R.C. Davis, and C.H. Coombs 159–165, New York, NY: John Wiley and Sons.

Debreu, Gérard. 1960. "Topological Methods in Cardinal Utility Theory." In *Mathematical Methods in the Social Sciences*, edited by Arrow, Kenneth J., Samuel Karlin, and Patrick Suppes 16–26, Stanford, CA: Stanford University Press.

Debreu, Gérard. 1964. "Continuity properties of Paretian utility." *International Economic Review* 5 (3): 285–293. 10.2307/2525513.

Kreps, David M. 1988. *Notes on the Theory of Choice.* Westview Press.

Kreps, David M. 2012. *Microeconomic Foundations I. Choice and Competitive Markets.* Princeton, NJ: Princeton University Press.

Mas-Colell, Andreu, Michael D. Whinston, and Jerry R. Green. 1995. *Microeconomic Theory.* Oxford University Press.

Rubinstein, Ariel. 2018. *Lectures in Microeconomic Theory.* Princeton University Press.