ECON0106: **Microeconomics**

Lecture Notes on Correspondences*

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These lecture notes first introduces the concepts of correspondences and their continuity, and then discuss two important results, Kakutani's fixed point theorem and Berge's maximum theorem.

1. Definitions

Definition 1. A **correspondence** F from X to Y is a set-valued function that associates every element in X a subset of Y, denoted by $F : X \rightrightarrows Y$ or $F : X \to 2^Y$, with $F(x) \subseteq Y$. For $A \subseteq X$, we define the image of F as $F(A) := \cup_{x \in A} F(x)$.
The set X is called the **domain** of the correspondence F, and Y is called the **codomain** of F. $F(x)$ is called the **image** of point $x \in X$.

You may consider the concept of correspondence as a generalisation of functions, in the sense that $F(x)$ is a set in Y instead of an element in Y. Clearly, a single-valued correspondence $F : X \rightrightarrows Y$ can be viewed as a function from X to Y.

Listed below are some terminologies that we use to describe the properties of correspondences.

Definition 2. A correspondence $F : X \rightrightarrows Y$ is said to be [property]-valued at $x_0 \in X$ if $F(x_0)$ is a [property] set. If F is [property]-valued at all $x_0 \in X$, we say F is [property]-valued.
This [property] can be

1. non-empty
2. single (singleton)
3. open
4. closed
5. compact
6. convex

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Notice that the 3 - 5 above requires Y to be a metric space (Y, d_Y), and 6 requires Y to be a (real) vector space $(Y, +, \cdot)$.

1.1. Upper Hemi-continuity

Similar to functions, it is possible to talk about continuity of a correspondence if its domain and codomain are both metric spaces. However, there are two distinct notions of continuity for correspondences, known as *upper hemicontinuity* and *lower hemicontinuity*, and they capture different aspects of continuity of a correspondence. Let's first look at upper hemicontinuity.

Definition 3. Let (X, d_X) and (Y, d_Y) be metric spaces. The correspondence $F : X \rightrightarrows Y$ is said to be **upper hemicontinuous (uhc) at** $x_0 \in X$ if $\forall$ open set U in (Y, d_Y) such that $F(x_0) \subseteq U$, there is $\delta > 0$ such that $F(B_\delta(x_0)) \subseteq U$.

The correspondence $F : X \rightrightarrows Y$ is said to be **upper hemicontinuous (uhc)** if it is upper hemicontinuous at x_0 for all $x_0 \in X$.

The definition requires that whenever the open set U covers the entire image of the point x_0, then it must also entirely cover all nearby images. What is not allowed by uhc at x_0 is sudden appearance of a chunk of image *outside of* U when x deviates from x_0.

For example, consider the correspondence $F_1 : \mathbb{R} \rightrightarrows \mathbb{R}$ defined as[1]

$$F_1(x) := \begin{cases} \{0\}, \text{ if } x \leq 0 \\ [-1, 1], \text{ if } x > 0 \end{cases}$$

Clearly F_1 fails to be uhc at 0, because if we let $U := (-1/2, 1/2)$, whenever x moves away a little from 0 to the right, the image $F_1(x)$ becomes $[-1, 1]$, which is not covered by U. The problem of this correspondence at 0 is that many new points suddenly appear when x deviate from 0 to the right, and this is a violation of uhc. Therefore, uhc can be intuitively interpreted as "no sudden appearance of a chunk of image when deviating from a point." (note this is stronger than uhc requires though)

Consider a slightly different correspondence $F_2 : \mathbb{R} \rightrightarrows \mathbb{R}$ defined as

$$F_2(x) := \begin{cases} \{0\}, \text{ if } x < 0 \\ [-1, 1], \text{ if } x \geq 0 \end{cases}$$

The image of F_2 at 0 is $[-1, 1]$, and so there is no sudden appearance of image when deviating from 0. Therefore, F_2 is uhc at 0. Clearly, F_2 is also uhc at all other points in $\mathbb{R}$, and so F_2 is uhc.

But note that this "no sudden appearance" tenet is just to provide some intuition for sufficient conditions: e.g. uhc allows "smooth changes" in the image when deviating from a point, if

[1] In $\mathbb{R}^n$, we use the Euclidean metric d_2 by default.

the correspondence is closed-valued at this point. For example, consider the correspondence $F_3 : \mathbb{R} \rightrightarrows \mathbb{R}$ defined as $F_3(x) := [x, x+1]$ for any $x \in \mathbb{R}$. Under F_3, the image $F_3(x) = [x, x+1]$ changes "smoothly" when x changes, and it can be shown that F_3 is uhc.

Claim 1. *The correspondence $F_3 : \mathbb{R} \rightrightarrows \mathbb{R}$ defined above is uhc.*

Proof. Take any $x_0 \in \mathbb{R}$. We focus on proving that F_3 is uhc at x_0.
Take any open set $U \supseteq [x_0, x_0+1]$. We want to show that $\exists\, \delta > 0$ such that $U \supseteq F(x)$ for any $x \in (x_0 - \delta, x_0 + \delta)$.
Because x_0 and $x_0 + 1$ are in the open set U, they are interior points of U, and so $\exists\, \delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subseteq U$ and $(x_0 + 1 - \delta, x_0 + 1 + \delta) \subseteq U$. Therefore, we have $(x_0 - \delta, x_0 + 1 + \delta) \subseteq U$. For any $x \in (x_0 - \delta, x_0 + \delta)$, we have $F(x) = [x, x+1] \subseteq (x_0 - \delta, x_0 + 1 + \delta) \subseteq U$. □

However, when the correspondence is not closed-valued, then even smooth changes in the image may violate uhc. For example, consider a slightly different correspondence $F_4 : \mathbb{R} \rightrightarrows \mathbb{R}$ defined as $F_4(x) := (x, x+1)$. It can be shown that it is not uhc at any point in $\mathbb{R}$. To see this, for each $x_0 \in \mathbb{R}$, let $U := F_4(x_0) := (x_0, x_0+1)$, and U cannot cover $F(x)$ as long as $x \neq x_0$. In applications, however, we almost always work with closed-valued correspondences, in which case uhc allows smooth changes, but does not allow sudden appearance of image.

For single-valued correspondences, uhc is equivalent to continuity of functions.

Proposition 1. *Let (X, d_X) and (Y, d_Y) be metric spaces. Consider a single-valued correspondence $F : X \rightrightarrows Y$. Define $f : X \to Y$ as $f(x) := y$ such that $y \in F(x)$. Then F is uhc at $x_0 \in X$ if and only if f is continuous at x_0.*

The proof is straightforward, and is left as an exercise.

For compact-valued correspondences, there is a *sequential characterisation of uhc*, which is formulated in the following proposition:[2]

Proposition 2. *Let (X, d_X) and (Y, d_Y) be metric spaces. Consider a correspondence $F : X \rightrightarrows Y$, and let $x_0 \in X$. Then the following two statements are equivalent:*

(1) F is compact-valued at x_0, and F is uhc at x_0.

(2) For any sequence (x_n) in X convergent to x_0, any sequence (y_n) such that $y_n \in F(x_n)$ for each $n \in \mathbb{N}$, there exists a subsequence $\left(y_{n_k}\right)$ convergent to some $y_0 \in F(x_0)$.

Proof. (1) $\Longrightarrow$ (2):
Take any sequence (x_n) in X convergent to x_0, any sequence (y_n) such that $y_n \in F(x_n)$ for each $n \in \mathbb{N}$. We want to show that there exists a subsequence $\left(y_{n_k}\right)$ convergent to some $y_0 \in F(x_0)$.

[2] This is the definition of uhc in the book by SLP, who only study compact-valued correspondences.

For each $k \in \mathbb{N}$, consider the set $U_k := \bigcup_{y \in F(x_0)} B_{1/k}(y)$. Because the arbitrary union of opens is still open, we know that U_k is an open set. By construction $U_k \supseteq F(x_0)$, and so by uhc of F at x_0, there exists $\delta_k > 0$ such that $F\left(B_{\delta_k}(x_0)\right) \subseteq U_k$. Since $x_n \to x_0$, there exists N_k such that $x_n \in B_{\delta_k}(x_0)$, and thus $y_n \in U_k$ for any $n > N_k$. Therefore, we can find a subsequence $\left(y_{n_k}\right)$ such that $y_{n_k} \in U_k$ for each $k \in \mathbb{N}$. By construction of U_k, for each k, there exists $z_k \in F(x_0)$ such that $d_Y\left(y_{n_k}, z_k\right) < 1/k$. As F is compact-valued at x_0, we know that $F(x_0)$ is compact in (Y, d_Y). So there exists a subsequence $\left(z_{k_l}\right)$ convergent to some $y_0 \in F(x_0)$. Therefore, we have $d_Y\left(z_{k_l}, y_0\right) \to 0$, and

$$\begin{aligned} 0 \le d_Y\left(y_{n_{k_l}}, y_0\right) &\le d_Y\left(y_{n_{k_l}}, z_{k_l}\right) + d_Y\left(z_{k_l}, y_0\right) \\ &< \frac{1}{k_l} + d_Y\left(z_{k_l}, y_0\right) \to 0 + 0 = 0 \end{aligned}$$

Consequently, we have $d_Y\left(y_{n_{k_l}}, y_0\right) \to 0$, which means $y_{n_{k_l}} \to y_0$. Finally, this means we have found a subsequence of (y_n) that converges to some point in $F(x_0)$.

(1) $\Longleftarrow$ (2):

(a) F is compact-valued at x_0.

Take any sequence (y_n) in $F(x_0)$. We will show that there exists a subsequence $\left(y_{n_k}\right)$ convergent to some $y_0 \in F(x_0)$.

Let $x_n = x_0$ for all $n \in \mathbb{N}$. Then we have $x_n \to x_0$ and $y_n \in F(x_n)$ for each $n \in \mathbb{N}$. By assumption, there exists a subsequence $\left(y_{n_k}\right)$ convergent to some $y_0 \in F(x_0)$.

(b) F is uhc at x_0.

Suppose that F is not uhc at x_0. Then $\exists\, U$ open in (X, d_X) such that $U \supseteq F(x_0)$, but $\forall \delta > 0$ we have $U \not\supseteq F(B_\delta(x_0))$. Hence, for any $n \in \mathbb{N}$, we have $U \not\supseteq F(B_{1/n}(x_0))$, i.e. there exists $x_n \in B_{1/n}(x_0)$ and $y_n \in F(x_n)$ such that $y_n \notin U$. Because $x_n \to x_0$, by assumption there exists a subsequence $\left(y_{n_k}\right)$ convergent to some $y_0 \in F(x_0)$.

Since $\left(y_{n_k}\right)$ is in $Y \setminus U$, which is closed in (Y, d_Y), we have $y_0 \in Y \setminus U$, and so $y_0 \notin F(x_0)$, a contradiction. □

Without compact-valuedness, uhc alone does not imply property (2) in the proposition above. For example, consider $F_5 : \mathbb{R} \rightrightarrows \mathbb{R}$ defined as $F_5(x) := (0, 1)$ for any $x \in \mathbb{R}$. Clearly, F_5 is uhc everywhere, but it does not satisfy property (2) at any $x_0 \in \mathbb{R}$, since compact-valuedness is necessary for property (2).

1.2. Closed Graph Property

There is a concept, called *closed graph property*, that is closely related to uhc.

Definition 4. Let (X, d_X) and (Y, d_Y) be metric spaces. The correspondence $F : X \rightrightarrows Y$ is said to have **closed graph property (cgp) at** $x_0 \in X$ if $\forall$ sequence (x_n) in X convergent to

x_0, $y_n \in F(x_n)$ for each $n \in \mathbb{N}$, and $y_n \to y_0 \in Y$, we have $y_0 \in F(x_0)$.
The correspondence $F : X \rightrightarrows Y$ is said to have **closed graph property (cgp)** if it has closed graph property at x_0 for all $x_0 \in X$.

Clearly, cgp implies closed-valuedness.

Claim 2. *Let (X, d_X) and (Y, d_Y) be metric spaces. If the correspondence $F : X \rightrightarrows Y$ is cgp at $x_0 \in X$, then it is closed-valued at x_0.*

Proof. Take any sequence (y_n) in $F(x_0)$ convergent to $y_0 \in Y$. We want to show that $y_0 \in F(x_0)$. Let $x_n = x_0$ for all $n \in \mathbb{N}$, then we have $x_n \to x_0$, $y_n \in F(x_n)$ for each $n \in \mathbb{N}$, and $y_n \to y_0 \in Y$. By cgp, we have $y_0 \in F(x_0)$. □

The **graph**[3] of a correspondence $F : X \rightrightarrows Y$ is defined as $Gr(F) := \{(x, y) \in X \times Y : y \in F(x)\}$

For a correspondence $F : X \rightrightarrows Y$, where (X, d_X) and (Y, d_Y) are metric spaces, the name of the property "closed graph property" comes from the fact that F has cgp (everywhere in X) if its graph is closed in $(X \times Y, d_{X\times Y})$, where the metric for the product space is defined as

$$d_{X\times Y}\left((x, y), \left(x', y'\right)\right) := \sqrt{[d_X(x, x')]^2 + [d_Y(y, y')]^2}$$

for any $(x, y), \left(x', y'\right) \in X \times Y$.

It can be shown that $d_{X\times Y}$ as defined above is a valid metric for $X \times Y$. Also, we can show that $(x_n, y_n) \to (x_0, y_0)$ in $(X \times Y, d_{X\times Y})$ if and only if $x_n \to x_0$ in (X, d_X) and $y_n \to y_0$ in (Y, d_Y), and this is left as an exercise.

Claim 3. *Let (X, d_X) and (Y, d_Y) be metric spaces. Then a correspondence $F : X \rightrightarrows Y$ has cgp if and only if $Gr(F)$ is closed in $(X \times Y, d_{X\times Y})$.*

Proof. $\Longrightarrow$:
Take any $((x_n, y_n))$ in $Gr(F)$ that is convergent to $(x_0, y_0) \in X \times Y$. We want to prove that $(x_0, y_0) \in Gr(F)$. As $(x_n, y_n) \to (x_0, y_0)$, we have that $x_n \to x_0$ and $y_n \to y_0$. Since $(x_n, y_n) \in Gr(F)$ for all n, we have $y_n \in F(x_n)$ for all n. As F has cgp, we know that F has cgp at x_0, and so $y_0 \in F(x_0)$, which implies $(x_0, y_0) \in Gr(F)$.
$\Longleftarrow$:
Take any $x_0 \in X$. We will show that F has cgp at x_0.
Take any (x_n) in X convergent to x_0, $y_n \in F(x_n)$ for each $n \in \mathbb{N}$, and $y_n \to y_0 \in Y$. We claim that $y_0 \in F(x_0)$.
Given that $x_n \to x_0$ and $y_n \to y_0$, we have $(x_n, y_n) \to (x_0, y_0)$ in $(X \times Y, d_{X\times Y})$. Because $y_n \in F(x_n)$ for each n, we have $(x_n, y_n) \in Gr(F)$ for each n. And since $Gr(F)$ is closed in $(X \times Y, d_{X\times Y})$, we have $(x_0, y_0) \in Gr(F)$. □

[3] This is in fact a redundant definition since $Gr(F) = F$, if we view F as a relation from $X \times Y$.

Closed graph property is closely related to uhc, and their relation is formulated by the following two propositions:

Proposition 3. *Let (X, d_X) and (Y, d_Y) be metric spaces. If a correspondence $F : X \rightrightarrows Y$ is uhc at $x_0 \in X$, and is closed-valued at x_0, then F has cgp at x_0.*

Proof. Take any sequence (x_n) in X convergent to x_0, $y_n \in F(x_n)$ for each $n \in \mathbb{N}$, and $y_n \to y_0 \in Y$.
We want to show that $y_0 \in F(x_0)$.
Suppose $y_0 \notin F(x_0)$, i.e. $y_0 \in Y \setminus F(x_0)$. Because F is closed-valued at x_0, $Y \setminus F(x_0)$ is open in (Y, d_Y), and so $\exists\, \epsilon > 0$ such that $B_{2\epsilon}(y_0) \subseteq Y \setminus F(x_0)$. And the "closed ball" $\bar{B}_\epsilon(y_0) := \{y \in Y : d_Y(y, y_0) \leq \epsilon\}$ is contained in $B_{2\epsilon}(y_0)$ and therefore in $Y \setminus F(x_0)$, and therefore $F(x_0) \subseteq Y \setminus \bar{B}_\epsilon(y_0)$. As a closed ball is a closed set and $F(x_0)$ is covered by the open set $Y \setminus \bar{B}_\epsilon(y_0)$. By uhc of F at x_0, $\exists\, \delta > 0$ such that $F(B_\delta(x_0)) \subseteq Y \setminus \bar{B}_\epsilon(y_0)$.

Given that $x_n \to x_0$ and $y_n \to y_0$, there exists $\hat{n}$ such that $x_{\hat{n}} \in B_\delta(x_0)$ and $y_{\hat{n}} \in \bar{B}_\epsilon(y_0)$. However, because $F(B_\delta(x_0)) \subseteq Y \setminus \bar{B}_\epsilon(y_0)$, we have $y_{\hat{n}} \in F(x_{\hat{n}}) \subseteq F(B_\delta(x_0)) \subseteq Y \setminus \bar{B}_\epsilon(y_0)$, which contradicts $y_{\hat{n}} \in \bar{B}_\epsilon(y_0)$. □

The result above states that uhc implies cgp if we have closed-valuedness. Without closed-valuedness, this implication does not hold since a uhc correspondence may not have closed-valuedness. For example, consider F_5 as previously defined. Clearly, F_5 is uhc everywhere, but it does not have cgp anywhere, since closed-valuedness is necessary for cgp.

A correspondence $F : X \rightrightarrows Y$, where (X, d_X) and (Y, d_Y) are metric spaces, is said to be **locally bounded at** x_0 if $\exists\, \delta > 0$ and a compact set K in (Y, d_Y) such that $F(B_\delta(x_0)) \subseteq K$. The next proposition works in the other direction:

Proposition 4. *Let (X, d_X) and (Y, d_Y) be metric spaces. If a correspondence $F : X \rightrightarrows Y$ has cgp at $x_0 \in X$, and F is locally bounded at x_0, then F is uhc at x_0.*

The proof of this proposition is similar to the proof of Proposition 2, part (b) of the direction "(1) $\Longleftarrow$ (2)".

Proof. Suppose that F is not uhc at x_0. Then $\exists U$ open in (Y, d_Y) such that $F(x_0) \subseteq U$, but $\forall \delta > 0$ we have $F(B_\delta(x_0)) \nsubseteq U$. Then for any $n \in \mathbb{N}$, we have $F(B_{1/n}(x_0)) \nsubseteq U$, i.e. there exists $x_n \in B_{1/n}(x_0)$ and $y_n \in F(x_n)$ such that $y_n \notin U$. By assumption there exists $\hat{\delta} > 0$ and compact set K in (Y, d_Y) such that $F(B_{\hat{\delta}}(x_0)) \subseteq K$. By construction, we have $x_n \to x_0$, and so $\exists N$ such that $x_n \in B_{\hat{\delta}}(x_0)$ and so $y_n \in K$ for any $n > N$.
By sequential compactness of K, there exists a subsequence (y_{n_k}) of $(y_n)_{n>N}$ convergent to some $y_0 \in K$. Because the subsequence $(y_{n_k}) \subseteq Y \setminus U$, which is closed, we have $y_0 \in Y \setminus U$.

However, since F has cgp at x_0, and $x_{n_k} \to x_0$, $y_{n_k} \in F(x_{n_k})$, $y_{n_k} \to y_0$, we have $y_0 \in F(x_0) \subseteq U$, a contradiction. □

The result above states that cgp implies uhc if we have local boundedness. Without local boundedness, cgp does not imply uhc. For example, consider $F_6 : \mathbb{R} \rightrightarrows [0,1)$ defined as

$$F_6(x) = \begin{cases} \{e^x\}, x < 0 \\ \{0\}, x \geq 0 \end{cases}$$

which is clearly not uhc at 0. However, F_6 has cgp at 0. Note that 1 is not in the codomain, and so when x_n converges to 0 from the negative real line, $y_n \in F(x_n)$ does not converge. This is not a violation of the proposition above, as F_6 is not locally bounded at 0. As 1 is not in the codomain, and so we cannot find a compact set K in $([0,1), d_2)$ to bound all images of points nearby 0.

Another example is $F_7 : \mathbb{R} \rightrightarrows \mathbb{R}$ defined as

$$F_6(x) = \begin{cases} \{1/x\}, x \neq 0 \\ \{0\}, x = 0 \end{cases}$$

As a consequence of the two propositions above, under closed-valuedness and local boundedness, uhc and cgp are equivalent.

1.3. Lower Hemi-continuity

Now let's define lower hemicontinuity.

Definition 5. Let (X, d_X) and (Y, d_Y) be metric spaces. The correspondence $F : X \rightrightarrows Y$ is said to be **lower hemicontinuous (lhc) at** $x_0 \in X$ if $\forall$ open set U in (Y, d_Y) such that $F(x_0) \cap U \neq \emptyset$, $\exists\, \delta > 0$ such that $F(x) \cap U \neq \emptyset$ for any $x \in B_\delta(x_0)$. The correspondence $F : X \rightrightarrows Y$ is said to be **lower hemicontinuous (lhc)** if it is lower hemicontinuous at x_0 for all $x_0 \in X$.

The definition requires that whenever the open set U covers a part of the image of the point x_0, then it must also cover a part of all nearby images. What is not allowed by lhc at x_0 is sudden disappearance of a chunk of image when x deviates from x_0.

For example, consider the correspondence $F_2 : \mathbb{R} \rightrightarrows \mathbb{R}$

$$F_2(x) := \begin{cases} \{0\}, \text{ if } x < 0 \\ [-1,1], \text{ if } x \geq 0 \end{cases}$$

as previously defined. Clearly F_2 fails to be lhc at 0, because if we let $U := (1/2, 3/2)$, whenever x moves away a little from 0 to the left, the image $F_2(x)$ becomes $\{0\}$, which does not share an intersection with U. The problem of this correspondence at 0 is that many points suddenly disappear when x deviate from 0 to the left, and this is a violation lhc. Therefore, lhc can be intuitively interpreted as "no sudden disappearance of a chunk of image when deviating from a point."

Consider the slightly different correspondence $F_1 : \mathbb{R} \rightrightarrows \mathbb{R}$

$$F_1(x) := \begin{cases} \{0\}, \text{ if } x \leq 0 \\ [-1,1], \text{ if } x > 0 \end{cases}$$

as previously defined. The image of F_1 at 0 is $\{0\}$, and so there is no sudden disappearance of image when deviating from 0. Therefore, F_1 is lhc at 0. Clearly, F_1 is also lhc at all other points in $\mathbb{R}$, and so F_1 is lhc.

Lower hemicontinuity does not allow sudden disappearance of image when deviating from a point, but it allows "smooth changes" in the image when deviating from a point. For example, consider the correspondence $F_3 : \mathbb{R} \rightrightarrows \mathbb{R}$

$$F_3(x) := [x, x+1]$$

for any $x \in \mathbb{R}$ as previously defined. Under F_3, the image $F_3(x) = [x, x+1]$ changes "smoothly" when x changes, and it can be shown that F_3 is lhc.

Claim 4. *The correspondence $F_3 : \mathbb{R} \rightrightarrows \mathbb{R}$ defined above is lhc.*

Proof. Take any $x_0 \in \mathbb{R}$. We want to show F_3 is lhc at x_0. Take any open set U for which $[x_0, x_0+1] \cap U \neq \emptyset$. We want to show that $\exists\, \delta > 0 : [x, x+1] \cap U \neq \emptyset$ for any $x \in (x_0 - \delta, x_0 + \delta)$. Let $\hat{x} \in [x_0, x_0+1] \cap U$. Because U is open, there exists $\delta > 0$ such that $(\hat{x} - \delta, \hat{x} + \delta) \subseteq U$. Take any $x \in (x_0 - \delta, x_0 + \delta)$. By construction, we have $x - x_0 \in (-\delta, \delta)$, and so $\hat{x} + (x - x_0) \in (\hat{x} - \delta, \hat{x} + \delta) \subseteq U$. As $\hat{x} \in [x_0, x_0 + 1]$, we have $\hat{x} + (x - x_0) \in [x_0 + (x - x_0), x_0 + (x - x_0) + 1] = [x, x+1]$. Therefore, we have $\hat{x} + (x - x_0) \in [x, x+1] \cap U$, and so $[x, x+1] \cap U \neq \emptyset$. □

Lower hemicontinuity allows smooth changes in the image, regardless of whether the correspondence is closed-valued. If we consider a slightly different correspondence $F_4 : \mathbb{R} \rightrightarrows \mathbb{R}$ defined as $F_4(x) := (x, x+1)$ for any $x \in \mathbb{R}$, a slightly modification of the proof above can show that F_4 is also lhc. For single-valued correspondences, lhc is equivalent to continuity of functions.

Proposition 5. *Let (X, d_X) and (Y, d_Y) be metric spaces. Consider a single-valued correspondence $F : X \rightrightarrows Y$. Define $f : X \to Y$ as $f(x) := y$ such that $y \in F(x)$. Then F is lhc at $x_0 \in X$ if and only if f is continuous at x_0.*

This proof is straightforward, and is left as an exercise.

The following proposition provides the *sequential characterisation of lhc*:

Proposition 6. *Let (X, d_X) and (Y, d_Y) be metric spaces. A correspondence $F : X \rightrightarrows Y$ is lhc at $x_0 \in X$ if and only if for any $y_0 \in F(x_0)$ and sequence (x_n) in X convergent to x_0, there exists $N \in \mathbb{N}$ and $y_n \in F(x_n)$ for any $n > N$ such that the sequence $(y_n)_{n>N}$ converges to y_0.*

In the proposition above, we start to construct the sequence (y_n) starting from $n = N + 1$, because $F(x_n)$ may be empty for small n's.

Proof. $\implies$:
Take any $y_0 \in F(x_0)$ and sequence (x_n) in X convergent to x_0. We want to prove that $\exists\, N \in \mathbb{N}$ and $y_n \in F(x_n)$ for any $n > N$ such that the sequence $(y_n)_{n>N}$ converges to y_0.
For each $k \in \mathbb{N}$, we have $y_0 \in F(x_0) \cap B_{1/k}(y_0)$, and so $F(x_0) \cap B_{1/k}(y_0) \neq \emptyset$. By lhc, $\exists\, \delta_k > 0$ such that for any $x \in B_{\delta_k}(x_0)$, we have $F(x) \cap B_{1/k}(y_0) \neq \emptyset$.
As $x_n \to x$, $\exists\, N \in \mathbb{N}$ such that $x_n \in B_{\delta_1}(x_0)$ for any $n > N$. For each $n > N$, arbitrarily take

$$y_n \in \bigcap_{k \in \left\{k' \in \mathbb{N} : x_n \in B_{\delta_{k'}}(x_0)\right\}} \left[F(x_n) \cap B_{1/k}(y_0)\right]$$

This is possible because $F(x_n) \cap B_{1/k}(y_0) \neq \emptyset$ whenever $x_n \in B_{\delta_k}(x_0)$.
Now we want to show that $(y_n)_{n>N}$ converges to y_0. Take any $\epsilon > 0$. $\exists\, K$ such that $1/k < \epsilon$ for any $k > K$. Since $x_n \to x_0$, $\exists\, \hat{N} > N$ such that $x_n \in B_{\delta_K}(x_0)$ for any $n > \hat{N}$. Therefore for any $n > \hat{N}$, we have $x_n \in B_{\delta_K}(x_0)$, which implies $y_n \in B_{1/K}(y_0)$, which in turn implies $y_n \in B_\epsilon(y_0)$.

$\impliedby$:
Suppose, by contradiction, that $\exists$ open set U in (Y, d_Y) such that $F(x_0) \cap U \neq \emptyset$, but $\forall\, \delta > 0$, $\exists\, x \in B_\delta(x_0)$ such that $F(x) \cap U = \emptyset$. This implies that for any $n \in \mathbb{N}$, $\exists\, x_n \in B_{1/n}(x_0)$ such that $F(x_n) \cap U = \emptyset$, i.e. $F(x_n) \subseteq Y \setminus U$.
By construction, we have $x_n \to x_0$. Arbitrarily take $y_0 \in F(x_0) \cap U$, and by assumption there exists $N \in \mathbb{N}$ and $y_n \in F(x_n)$ for any $n > N$ such that the sequence $(y_n)_{n>N}$ converges to y_0. And given that $y_n \in F(x_n) \subseteq Y \setminus U$ for any $n > N$, and $Y \setminus U$ is closed in (Y, d_Y) since U is open, we have $y_0 \in Y \setminus U$. This contradicts the construction of y_0. □

As we have discussed, uhc for closed-valued correspondences means no sudden appearance of image when deviating from a point, while lhc means no sudden disappearance of image when deviating from a point. Therefore, we might expect a relation between F being uhc and F^c being lhc. In fact, we have one direction, but not the other.

For a correspondence $F : X \rightrightarrows Y$, let's define its complement $F^c : X \rightrightarrows Y$ as

$$F^c(x) := Y \setminus F(x)$$

for any $x \in X$. (This is a redundant definition if we realise that F is essentially a subset of $X \times Y$.)

Proposition 7. *Let (X, d_X) and (Y, d_Y) be metric spaces, and consider a correspondence $F : X \rightrightarrows Y$. If F^c is uhc at $x_0 \in X$, then F is lhc at x_0.*

The proof is left as an exercise.

However, F^c being lhc does not imply F being uhc, even if we further assume F to be compact-valued. For example, consider the correspondence $F_7 : \mathbb{R} \rightrightarrows \mathbb{R}$ defined as:

$$F_8(x) := \begin{cases} \{0\}, \text{ if } x < 0 \\ \{1\}, \text{ if } x \geq 0 \end{cases}$$

Clearly F is compact-valued, and $F(x)$ is not uhc at 0. However, F^c is lhc at all $x_0 \in \mathbb{R}$.

Finally, a correspondence is said to be continuous if it is both uhc and lhc.

Definition 6. Let (X, d_X) and (Y, d_Y) be metric spaces. The correspondence $F : X \rightrightarrows Y$ is said to be **continuous at** $x_0 \in X$ if F is both uhc and lhc at x_0. The correspondence F is said to be **continuous** if F is continuous at x_0 for all $x_0 \in X$.

2. Kakutani's Fixed Point Theorem

Definition 7. A correspondence F from X to X itself is called a **self-correspondence**. For a self-correspondence $F : X \rightrightarrows X$, a point $x \in X$ is called a **fixed point** of F if $x \in F(x)$.

When the self-correspondence F is single-valued, clearly $x \in X$ is a fixed point of F if $F(x) = \{x\}$, which is consistent with notion of fixed points for functions. Therefore, the definition above can be considered as a generalisation of the notion of fixed points to correspondences.

Theorem 1. ***(Kakutani's Fixed Point)*** *Let X be a nonempty, compact, and convex set in $\mathbb{R}^n$. If the self-correspondence $F : X \rightrightarrows X$ is nonempty-valued, compact-valued, convex-valued, and uhc, then there exists a fixed point $x \in X$ of F.*

In the theorem above, compactness is with respect to the metric space $(\mathbb{R}^n, d_2)$, and convexity is with respect to the vector space $(\mathbb{R}^n, +, \cdot)$ over $\mathbb{R}$, where $+$ and $\cdot$ are the usually defined vector addition and scalar multiplication for real vectors.

If F is single-valued, then nonempty-valuedness, compact-valuedness, and convex-valuedness of F holds trivially, and uhc reduces to the continuity of functions. So the theorem above reduces to Brouwer's fixed point theorem. Therefore, Kakutani's fixed point theorem should be viewed as a generalisation of Brouwer's fixed point theorem.

Because the codomain X of F is compact in the theorem, compact-valuedness is equivalent to closed-valuedness, and so we can replace the compact-valuedness assumption by closed-valuedness.

Again because the codomain X is compact, (compact-valuedness + uhc) is equivalent to cgp. To see this, the direction " $\Longrightarrow$ " is given by Proposition 3, and the other direction " $\Longleftarrow$ " is given by Proposition 4, since local boundedness holds trivially. Therefore we have the following corollary.

Corollary 1. *Let X be a nonempty, compact, and convex set in $\mathbb{R}^n$. If the self-correspondence $F : X \rightrightarrows X$ is nonempty-valued, convex-valued, and has cgp, then there exists a fixed point $x^* \in X$ of F.*

Kakutani's fixed point theorem plays the central role in the proof of the existence of Nash equilibria in non-cooperative game theory.

3. Berge's Theorem of Maximum

Theorem 2. ***(Berge's Maximum Theorem)*** *Let X and Θ be metric spaces, $f : X \times \Theta \to \mathbb{R}$ be a continuous function, and $B : \Theta \rightrightarrows X$ be a non-empty and compact-valued correspondence. Let $f^*(\theta) := \sup_{x \in B(\theta)} f(x,\theta)$ and $X^*(\theta) := \operatorname{argmax}_{x \in B(\theta)} f(x,\theta)$. If B is continuous at $\theta \in \Theta$, then f^* is continuous at θ and X^* is uhc, nonempty, and compact-valued at θ.*

In the theorem above, the maximisation problem we are looking at is a parameterised problem

$$\max_{x \in X} f(x,\theta) \text{ such that } x \in B(\theta)$$

where both the objective function f and the constraint set B depend on the parameter θ. The theorem states that if the objective function f is continuous, and the constraint set B is nonempty- and compact-valued, and is both uhc and lhc in the parameter θ, then the set of maximisers X^* is compact and uhc in θ, and the maximum value f^* is also continuous in θ.

Proof. Let's prove the theorem in three steps:

Step 1: X^* is nonempty-valued at θ_0
As f is jointly continuous in (x,θ), then $f(\cdot,\theta)$ is continuous in x. As we also have that B is nonempty- and compact-valued at θ_0, the claim follows from Weierstrass extremum theorem.

Step 2: X^* is compact-valued and uhc at θ_0
We prove this using Proposition 2. Take any sequence (θ_n) in Θ convergent to θ_0, any sequence (x_n) such that $x_n \in X^*(\theta_n)$ for each $n \in \mathbb{N}$.
We now prove that $\exists$ subsequence $\left(x_{n_k}\right)$ convergent to some $x_0 \in X^*(\theta_0)$. Given that $x_n \in X^*(\theta_n) \subseteq B(\theta_n)$ for each n, and as B is compact-valued and uhc at θ_0, by Proposition 2, $\exists$ subsequence $\left(x_{n_k}\right)$ convergent to some $x_0 \in B(\theta_0)$.
Take the x_0 found this way; it is sufficient to show that $x_0 \in X^*(\theta_0)$, i.e. $f(x_0,\theta_0) \geq f(z,\theta_0)$ for any $z \in B(\theta_0)$. Because B is lhc at θ_0 and $\theta_{n_k} \to \theta_0$, by sequential definition of lhc (Proposition 6), there exists $K \in \mathbb{N}$ and $z_k \in B\left(\theta_{n_k}\right)$ for each $k > K$, such that $z_k \to z$. As $x_{n_k} \to x_0$, $z_n \to z$, $\theta_{n_k} \to \theta_0$, by continuity of f, $f\left(x_{n_k},\theta_{n_k}\right) \to f(x_0,\theta_0)$ and $f\left(z_k,\theta_{n_k}\right) \to f(z,\theta_0)$. For each k, we have $f(x_{n_k},\theta_{n_k}) \geq f(z_k,\theta_{n_k})$ because $x_{n_k} \in X^*\left(\theta_{n_k}\right)$. Therefore we have $f(x_0,\theta_0) \geq f(z,\theta_0)$.

Step 3: f^* is continuous at θ_0

Let us show the continuity of f^* at θ_0 by using the sequential definition of continuous functions.

Take any sequence (θ_k, x_k) such that $\theta_k \to \theta_0$ and $x_k \in X^*(\theta_k)$.[4] Since X^* is compact-valued and uhc at θ_0, by Proposition 2, there exists a subsequence (x_l) of (x_k) convergent to some point $x_0 \in X^*(\theta_0)$. Because $x_l \in X^*(\theta_l)$ and $x_0 \in X^*(\theta_0)$, we have $f(x_l, \theta_l) = f^*(\theta_l)$ and $f(x_0, \theta_0) = f^*(\theta_0)$. Then, as $\theta_l \to \theta_0$, $x_l \to x_0$, we have $(x_l, \theta_l) \to (x_0, \theta_0)$, and so $f^*(\theta_l) = f(x_l, \theta_l) \to f(x_0, \theta_0) = f^*(\theta_0)$.

To see that this implies continuity of f^*, take any sequence θ_n convergent to θ_0 and suppose that $f^*(\theta_n) \not\to f^*(\theta_0)$. Then, there is $\epsilon > 0$ and a subsequence (θ_k), $\{\theta_k\}_k \subseteq \{\theta_n\}_n$, such that $|f^*(\theta_k) - f^*(\theta_0)| \geq \epsilon$ for every k. Using the above result, we get that for a subsequence (θ_l) of (θ_k), $f^*(\theta_l) \to f^*(\theta_0)$, and we obtain a contradiction. $\square$

By Berge's maximum theorem, we can only conclude that the set of maximisers X^* is uhc in the parameter θ. In fact, X^* may easily fail to be lhc, even when the objective function f and the constraint correspondence B are continuous in the parameter θ. For example, consider the following problem:

$$\max_{(x_1,x_2)\in\mathbb{R}^2_+} \theta x_1 + x_2 \text{ such that } p_1 x_1 + p_2 x_2 \leq w$$

where the parameters $\theta > 0, p_1, p_2, w > 0$. Clearly, the objective function $f : \mathbb{R}^2_+ \times \mathbb{R}_+$ defined as

$$f(x, \theta, p_1, p_2, m) := \theta x_1 + x_2$$

is continuous. The constraint correspondence $B : \mathbb{R}_+ \times \mathbb{R}^2_{++} \times \mathbb{R}_{++} := S \rightrightarrows \mathbb{R}^2_+$ defined as $B((p_1, p_2), w; \theta) := \left\{x \in \mathbb{R}^2_+ : p_1 x_1 + p_2 x_2 \leq w\right\}$ is nonempty- and compact-valued, and continuous at all $(\theta, p_1, p_2, w) \in S$. Therefore the assumptions of Berge's maximum theorem are satisfied. However, it is not difficult to see that the set of maximisers $X^* : \mathbb{R}_+ \rightrightarrows \mathbb{R}^2_+$ is given by

$$X^*((p_1, p_2), w; \theta) := \begin{cases} \left\{\left(0, \frac{w}{p_2}\right)\right\}, \text{ if } p_1 > \alpha p_2 \\ \left\{x \in \mathbb{R}^2_+ : p_1 x_1 + p_2 x_2 = w\right\}, \text{ if } p_1 = \theta p_2 \\ \left\{\left(\frac{w}{p_1}, 0\right)\right\}, \text{ if } 0 < p_1 < \theta p_2 \end{cases}$$

which is clearly uhc but not lhc at the point $(\theta, (p_1, p_2), w)$ where $p_1 = \theta p_2$.

If, in addition to the assumptions in the Theorem 2, B is convex-valued and f is (strictly) quasi-concave in x, then X^* is convex (resp. single-valued). In the case when X^* is single-valued, we can think of X^* as a continuous function x^*, such that $X^*(\theta) = \{x^*(\theta)\}$.

[4]Here is where we are using the fact that B is nonempty- and compact-valued and $f(\cdot, \theta)$ is continuous to obtain that $X^*(\theta)$ is nonempty. If we have that for all sequences (θ_n) converging to θ_0 we have $X^*(\theta_n) \neq \emptyset$, or if we restrict the domain of f^* to the θ such that $X^*(\theta) \neq \emptyset$, then we just need B to be compact-valued at θ_0 to show continuity of f^*.