ECON0106: **Microeconomics**

4. Monotone Comparative Statics of Individual Choices*

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1. Overview

As the previous lectures have shown, a typical problem in economics regards constrained optimisation, where an agent choose and action x from $S \subseteq X$ to maximise an objective function $f : X \to \mathbb{R}$. It is then of interest to understand how the agent's behaviour, as given by

$$X(S;f) := \arg\max_{x \in S} f(x),$$

changes when their objective f or their feasible set S change. This is what is typically termed comparative statics.

Comparative statics are monotone when one makes claims that $X(S;f)$ "increases" in some sense when S or f also "increase." A canonical example is that — fixing output level — firm demand for an input decreases weakly in the price to that input. A related question is whether — fixing input prices — firm demand for inputs increases in the target output. Further, when there multiple cost-minimising manners to organise production, how should we compare the two sets of optimal inputs?

A classical approach to this problem is to consider the use of calculus, relying on the Lagrangian and a calculus-based version of the envelope theorem. In this lecture we will learn how to be able to derive comparative statics results *without* using calculus.

2. General Definitions

We will start by defining a way to order elements. Let $(X, \geq)$ be a **partially ordered set**, that is, $\geq$ is a binary relation on X that is reflexive, transitive, and anti-symmetric.

Two important notational definitions that are easy to mistake are those of a join ($\vee$) and a meet ($\wedge$). The **join** of two elements x, x' taken with respect to X, written $x \vee_X x'$ corresponds to the $\geq$-smallest element in X that is simultaneously larger than both x and x': $x \vee_X x' := \inf\{y \in X : y \geq x \text{ and } y \geq x'\}$, where the infimum is taken with respect to $\geq$. The **meet** — denoted by

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$x \wedge_X x'$ — is symmetrically defined: the $\geq$-largest element in X that are simultaneously smaller than both x and x': $x \wedge_X x' := \sup\{y \in X : x \geq y \text{ and } x' \geq y\}$.

We then call $(X, \geq)$ different names depending on the properties it satisfies regarding joins and meets:

Definition 1. (i) A partially ordered set $(X, \geq)$ for which joins and meets exist for any pair of elements — i.e. $\forall x, x' \in X$, $x \vee_X x' \in X$ and $x \wedge_X x' \in X$ — is called a **lattice**.

(ii) A **complete lattice** is one where any subset attains its supremum and infimum in the set: $\forall S \subseteq X$, $\sup_X S \in X$ and $\inf_X S \in X$[1]

(iii) A sublattice of X is a subset $S \subseteq X$ where that includes the joins and the meets of any two of its elements, where the joins and meets are taken in X, i.e. S is a **sublattice** if $\forall x, x' \in S$, $x \vee_X x' \in S$ and $x \wedge_X x' \in S$.

(iv) A **complete sublattice** S is a sublattice of X for which the supremum and infimum of any of its subsets $S' \subseteq S$ is contained in S. That is, a sublattice is complete if any of its subsets attains its supremum and infimum in it — again the supremum and infimum are taken in X.

Below are some examples that can help gaining intuition:

Example 1. 1. $((0,1), \geq)$ is a lattice but not a complete lattice.

2. $(\mathbb{R}^k, \geq)$, where $\geq$ is the natural product order[2] is a lattice.
For any sublattice $S \subseteq \mathbb{R}^k$, S is a complete sublattice if and only if it is compact; further $(S, \geq)$ is then also a complete lattice (Topkis, 1998, Theorem 2.3.1.).

3. $(0,1) \subseteq \mathbb{R}$ is a sublattice of $(\mathbb{R}, \geq)$ but not a complete sublattice, where $\geq$ is the natural order.

4. Under the natural product order, $\{(0,0), (1,0), (0,1), (2,2)\}$ is a complete lattice, but not a sublattice of $\mathbb{N}^2$.

5. Under the natural product order, $\{(0,0), (1,0), (0,1), (2,1), (1,2), (2,2)\}$ is not a lattice.

3. Ordering Sets

As mentioned earlier, one of the main complications is how to order sets based on the given partial order $\geq$. The existing literature provides some different alternatives. The most conventional one is the **strong set order** $\geq_{ss}$ (Topkis, 1979, 1998; Milgrom and Shannon, 1994), where $\geq_{ss}$ is a binary relation on the powerset of some set X, 2^X. It is defined as follows:

Definition 2. We say that S' **strong set dominates** S (writing $S' \geq_{ss} S$) if $\forall x' \in S', x \in S$, $x \vee x' \in S'$ and $x \wedge x' \in S$.

[1] Sometimes you will also see the notation $\vee_X S$ and $\wedge_X S$ instead of $\sup_X S$ and $\inf_X S$.
[2] $x \geq y$ if $x_i \geq y_i$ for every $i \in [k]$.

That is, a set S strong set dominates another set S' if, taking any one element from each set, their join belongs to the dominating set and their meet to the dominated set.

Exercise 1. *For instance, recall our definition of budget sets* $B(p,w) := \{x \in \mathbb{R}^k_+ \mid p \cdot x \leq w\}$*, with* $p \in \mathbb{R}^k_{++}$ *and* $w \in \mathbb{R}_+$*. Suppose that* $\succsim$ *are strongly monotone and assume* $k \geq 2$*.*

(i) Prove that $\neg(B(p,w) \geq_{ss} B(p,w))\ \forall w > 0$*.*

(ii) Fix p *and provide necessary and sufficient conditions on* w', w *so that* $B(p,w') \geq_{ss} B(p,w)$*.*

The strong set order can be too demanding and therefore inapplicable to many situations; this is one possible motivation for the weak set order (Che et al., 2021) that we will study later on.

4. Ordering Functions

We now want good notions to compare functions. And the supply more than met demand.

Definition 3. Let f be a real-valued function on $X \times T$, where X, T are partially ordered sets, and joins and meets of elements in $X \times T$ are with respect to the product order. We say that

(i) f satisfies the **single-crossing property** (SCP) in $(x;t)$ if $\forall x, x' \in X$, $t, t' \in T$, such that $x' > x$ and $t' > t$, $f(x';t) - f(x;t) \geq (>) 0 \implies f(x';t') - f(x;t') \geq (>) 0$. It satisfies the ***strict*** **single-crossing property** if the last inequality is always strict.

(ii) f has **increasing differences** (ID) in $(x;t)$ if $\forall x, x' \in X$, $t, t' \in T$, such that $x' > x$ and $t' > t$, $f(x';t') - f(x;t') \geq f(x';t) - f(x;t)$. It has ***strict*** **increasing differences** if the last inequality is always strict.

(iii) f is **quasisupermodular** (QSM) in (x,t) if $\forall y, y' \in X \times T$, $f(y) - f(y \wedge y') \geq (>) 0 \implies f(y \vee y') - f(y') \geq (>) 0$.

(iv) f is **supermodular** (SM) in (x,t) if $\forall y, y' \in X \times T$, $f(y \vee y') - f(y') \geq f(y) - f(y \wedge y')$. f is **submodular** if $-f$ is supermodular.

Note that SM $\implies$ {QSM, ID} $\implies$ SCP, i.e. satisfying the single-crossing property in $(x;t)$ is weaker than satisfying quasisupermodularity or increasing differences and each of these is weaker than satisfying supermodularity.

Single-crossing and quasisupermodularity provide *ordinal* conditions on f that can be readily translated into restrictions on preference relations. In contrast, increasing differences and supermodularity are their respective *cardinal* counterparts. It is then very much surprising that Chambers and Echenique (2009) found that preference relations on a lattice have a weakly monotone and quasisupermodular utility representation if and only if they have a weakly monotone and supermodular utility representation.

Some useful properties of supermodular functions:

Exercise 2. *Prove the following statements:*

(i) If f and g are supermodular real-valued functions on X, then $\alpha f + \beta g$ are supermodular $\forall \alpha, \beta \geq 0$.

(ii) If $\exists$ strictly increasing $g : \mathbb{R} \to \mathbb{R}$ such that $g \circ f$ is supermodular, then f is quasisupermodular.

(iii) If $f \in \mathscr{C}^2$ in $y \in Y \equiv X \times T$, then f is supermodular in y if and only if $\frac{\partial^2}{\partial y_i \partial y_j} f \geq 0$, $\forall i \neq j$.

(iv) If X and Y are partially ordered sets, $X \times Y$ is a lattice with respect to the product order, and $f : X \times Y \to \mathbb{R}$ is supermodular, then $g(x) := \sup_{y \in Y} f(x, y)$ is supermodular.

The above are properties of *a* function. But note that given we have $f(x, t)$, we can interpret it as a parametrised family of functions $f_t(x) := f(x, t)$. So, we adjust the above definitions to handle comparison of functions. We will focus on one of them, single-crossing.

Definition 4. Let v, u be two real-valued functions on X; v **single-crossing** dominates u ($v \geq_{sc} u$) if $\forall x, x' \in X$ such that $x' \geq x$, $u(x') - u(x) \geq (>) 0 \implies v(x') - v(x) \geq (>) 0$.

5. Monotone Comparative Statics of Individual Choices

5.1. Strong Monotone Comparative Statics

Theorem 1. *(Monotonicity; (Milgrom and Shannon, 1994, Theorem 4)) Let X be a lattice and v, u be two real-valued functions on X. v and u are quasisupermodular and v single-crossing dominates u if and only if, for $S' \geq_{ss} S$, $X(S'; v) \geq_{ss} X(S; u)$.*

Proof. $\implies$: Take any $x \in X(S; u), x' \in X(S'; v)$. As $S' \geq_{ss} S$, we have $x \wedge x' \in S$ and $x \vee x' \in S'$. Then

$$
\begin{aligned}
& x \in X(S; u) && \\
\implies & u(x) - u(x \wedge x') \geq 0 && \text{optimality of } x \\
\implies & u(x \vee x') - u(x') \geq 0 && \text{quasisupermodularity of } u \\
\implies & v(x \vee x') - v(x') \geq 0 && v \geq_{sc} u \\
\implies & x \vee x' \in X(S'; v) && \text{optimality of } x';
\end{aligned}
$$

and

$$\begin{aligned}
& x' \in X(S';v) & \\
\implies & v(x \vee x') - v(x') \le 0 & \text{optimality of } x' \\
\implies & v(x) - v(x \wedge x') \le 0 & \text{quasisupermodularity of } v \\
\implies & u(x) - u(x \wedge x') \le 0 & v \ge_{sc} u \\
\implies & x \wedge x' \in X(S;u) & \text{optimality of } x.
\end{aligned}$$

Hence $X(S';v) \ge_{ss} X(S;u)$.

$\Longleftarrow$:

To show necessity of quasisupermodularity, let $S = \{x, x \wedge x'\}$, $S' = \{x', x \vee x'\}$, $\neg(x' \ge x)$, and $u = v$. Clearly, $S' \ge_{ss} S$. Note that if we have $u(x) \ge (>) u(x \wedge x') \iff x \in (=) X(S;u)$. As $X(S';u) \ge_{ss} X(S;u)$, then $x \in (=) X(S;u) \implies x \vee x' \in (=) X(S';u) \implies u(x \vee x') \ge (>) u(x')$.
To show necessity of single-crossing, let $S = \{x, x'\}$ with $x' > x$. As $X(S;v) \ge_{ss} X(S;u)$, $x' \in (=) X(S;u) \implies x' \in (=) X(S;v)$. And then, $u(x') - u(x) \ge (>) 0 \implies v(x') - v(x) \ge (>) 0$. □

Some other results that are easy to obtain by adjusting the proof above:

Corollary 1. *(Milgrom and Shannon, 1994, Corollary 1) Let X be a lattice and f a real-valued function on X. f is quasisupermodular if and only if, for $S' \ge_{ss} S$, $X(S';f) \ge_{ss} X(S;f)$.*

Corollary 2. *(Milgrom and Shannon, 1994, Corollary 2) Let X be a lattice, S a sublattice, and f a real-valued function on X. If f is quasisupermodular, then $X(S;f)$ is a sublattice of S.*

Corollary 3. *(Monotone Selection; Milgrom and Shannon 1994, Theorem 4') Let X be a lattice, v, u be two real-valued functions on X, and $S' \ge_{ss} S$, with $S, S' \subseteq X$. If v and u are quasisupermodular and v strictly single-crossing dominates u, then $\forall x' \in X(S';v), x \in X(S;u)$, $x' \ge x$.*

This last result is stronger than it might look like at first glance: it is saying that *any* maximiser in $X(S';v)$ is greater than *any* maximiser in $X(S;u)$.

This next two exercises will guide you through how to apply these results:

Exercise 3. *Suppose a firm hires labor $l \in \mathbb{R}_+$ for a wage rate $w > 0$ and capital $k \in \mathbb{R}_+$ for $r > 0$, in order to produce a quantity $y \in \mathbb{R}_+$. The firm sells its product at a price $p > 0$. Prices p, r, w are taken as given. Their production function, mapping combinations of inputs to output quantities, is given by $F : \mathbb{R}^2_+ \to \mathbb{R}_+$, and we write $y = F(k,l)$. Their profit maximisation problem is then*

$$\max_{(k,l) \ge 0} \pi(k,l;p,r,w) = pF(k,l) - rk - wl.$$

Throughout we fix (and omit dependence on) (p,w), and denote the optimal levels of inputs by $k^*(r)$ and $l^*(r)$

(i) Show that $\pi(k,l;r)$ has strict increasing differences in $(-k, r)$ and strict single-crossing prop-

erty in $(-k,r)$.

(ii) Show that $\bar{l}(k,r) := \arg\max_{l\geq 0} \pi(k,l;r)$ *may depend on* k *but does not depend (directly) on* r*; we can then write* $\bar{l}(k)$ *instead of* $\bar{l}(k,r)$.

(iii) Assume $\bar{l}$ *is a function (or fix a selection) and prove that* $\pi(k,\bar{l}(k);r)$ *has strict increasing differences in* $(-k,r)$ *and strict single-crossing property in* $(-k,r)$. *Use this to show that* $k^*(r) = \arg\max_{k\geq 0} \pi(k,\bar{l}(k);r)$ *is nonincreasing in* r. *Conclude that the own-price effect for capital is negative on the firm's capital demand.*

(iv) Now let's consider the effect of a change in r *on* l. *Assume* $k^*(r)$ *is a function (or fix a selection) and define* $l^*(r) := \arg\max_{l\geq 0} \pi(k^*(r),l;r)$. *Show that if* $F(k,l)$ *has increasing differences in* (k,l) *(resp.* $(-k,l)$*), then (i)* $\pi(k^*(r),l;r)$ *has increasing differences in* $(l,-r)$ *(resp.* (l,r)*) and (ii) conclude* $l^*(r)$ *is nonincreasing (resp. nondecreasing) in* r*, i.e. that increasing the price of capital weakly decreases the optimal level of capital and consequently weakly decreases (increases) the amount of labor hired.*

(v) Comment on the relation between increasing differences in (k,l) *vs.* $(-k,l)$ *and factor complementarity/substitutability.*

Exercise 4. *Suppose Robinson Crusoe is stranded on a desert island with a supply* $e > 0$ *of seedcorn. He will be rescued two years from now (and he knows this), so his problem is how to allocate the* e *units of seedcorn between current consumption and planting for next year's consumption. He can plant* $x \in [0,e]$ *units of corn, consume* $e - x$ *this year, and get a crop of* $f(x)$ *next year, where* f *is nondecreasing and satisfies* $f(0) = 0$. *His total utility is given by* $U(e-x, f(x))$ *and his problem is then given by* $\max_{x\in[0,e]} U(e-x, f(x))$.

Let $x^*(e)$ *be the set of maximisers for a given* e.

(i) Show that $x^*(e)$ *is a convex set if* U *is quasi-concave,* f *concave, and* U *is increasing in its second argument.*

(ii) Show that $x^*(e)$ *is a singleton if* U *is quasi-concave,* f *strictly concave, and* U *strictly increasing in its second argument.*

(iii) Assume that U *and* f *are twice continuously differentiable, strictly concave, and strictly increasing. Use monotone comparative statics to show that* x^* *is nondecreasing in* e.

6. Further Reading

Standard References: While these are not new ideas, I don't know any economics textbook that provides a good treatment (if you do, please let me know!). A useful guideline — aside from the references used — might be Federico Echenique's comprehensive notes on the topic (https://eml.berkeley.edu/~fechenique/lecture_notes/echenique_MCS.pdf).

Related questions/topics: Whatever your model may be, you want to be able to say something general like if A changes in this way, then B changes in that way. This makes comparative statics are bread-and-butter (or jam, if you prefer) of results in economic theory.

The ideas here have been extended to choice under uncertainty (see Athey, 2002), and they have been used in many applications, including IO and macroeconomics. Later on, we'll see some general monotone comparative statics results for equilibria. I think that comparative statics results can also be particularly relevant for empirical work, as not only do they correspond, in essence, to counterfactuals, but also because they can be leveraged for identification strategies.

References

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