ECON0106: **Microeconomics**

6. Risk Attitudes*

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1. Overview

In this lecture, we will introduce and study risk attitudes. Attitudes toward risk are of fundamental importance in understanding individuals' behaviour in face of risk: how they constitute their financial portfolio, their behaviour in the context of a pandemic, their purchasing decisions, their willingness to take up a job or continue searching for a better one, and they even relate to how people vote. By grounding our definitions in terms of primitives, we not only gain a better understanding on what properties of expected utility representations imply, we are also then able to identify and test statements based on data.

We will focus on the case where the decision-maker has preferences over gambles affecting their wealth. Why wealth? Well, this can be motivated by making use of, for instance, consumer theory. As we have seen, with continuous preferences on the set of bundles, $v(p,w) = \max_{x \in B(p,w)} u(x)$. With continuous preferences satisfying independence and the Archimedean property on the set of distributions over wealth, we get a utility representation that looks like $\mathbb{E}_F[v(p,\cdot)]$. And, with local nonsatiation, we also get that $v(p,\cdot)$, our Bernoulli utility, is strictly increasing in w.[1]

Our first task is to introduce and study behavioural notions of risk aversion, that is, definitions which can be falsified with data; in our case, choice data. After defining these, we show how these relate to properties of the Bernoulli utility function in an expected utility representation. We then provide a behavioural way to compare individuals in terms of their risk attitudes, even if they are not risk averse, and again show how this related to strutural properties of their expected utility representations. Finally, we consider how attitudes toward risk can be affected by wealth.

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[1]For details on induced preferences on wealth, see Kreps (2012, Section 6.2).

2. Setup

Our agents will now have expected utility representations not just on probability distributions over a finite set of items, but over the real line.

Let X be a convex subset of $\mathbb{R}$ where $x \in X$ denotes the final wealth of the decision-maker (the outcome). Probabilities are now given by (cumulative) distributions functions $F : \mathbb{R} \to [0,1]$ such that F is nondecreasing, right-continuous, $\lim_{x\to-\infty} F(x) = 0$, and $\lim_{x\to\infty} F(x) = 1$ with support on X, i.e. $\mathbb{P}_F(X) = \int_X dF(x) = 1$. For any (cumulative) distribution function F, we denote the expectation operator by $\mathbb{E}_F[\cdot]$ and we define its mean as $\mu_F := \int_X x dF(x)$. We will focus on the set of distributions on X with finite mean, which we denote by $\mathscr{F}$.

We want $\succsim$ to have expected utility representation, that is, a $u : X \to \mathbb{R}$ such that $\forall F, G$, $F \succsim G \iff \mathbb{E}_F[u] \geq \mathbb{E}_G[u]$. For convenience, we will define $U(F) := \mathbb{E}_F[u]$. In order to make sure that we have a utility representation U, we will assume that $\succsim$ is a continuous preference relation on $\mathscr{F}$. For us to have an expected utility representation u, $\succsim$ has to also satisfy independence and the Archimedean property.[2] Note that this implies that $U(F)$ has to be finite, as otherwise it will not satisfy the Archimedean property. Finally, we will assume that the decision-maker prefers more money to less, i.e. $x > y \implies \delta_x \succ \delta_y$ ($\succsim$ is strictly monotone in $\{\delta_x, x \in X\}$). This will result in having u be strictly increasing.

To simplify the statement of the results, we set this as an assumption that we maintain throughout this lecture:

Assumption 1. The preference relation $\succsim$ on $\mathscr{F}$ has an expected utility representation with $u : X \to \mathbb{R}$ strictly increasing.

3. Risk Attitudes

We want a concept that captures the idea of avoiding and seeking risk; this is main motivation for the definitions of risk averse/seeking preferences. For instance, an intuitive definition of a person who is risk averse is someone who would decline to take fair gambles, say making or losing £1 with equal probability. Naturally, risk seeking would be defined as the opposite, and risk neutral as someone who is indifferent between taking and not taking the fair bet. Then, if you extend this notion to lotteries that do not have expected value of zero, you would have that a risk averse person is someone who prefers to get the expected value of a lottery to taking the lottery. This is exactly what our definition is saying:

Definition 1. A preference relation $\succsim$ on $\mathscr{F}$ is

[2] We are glossing over some subtleties here; see section 5.2 in Kreps (2012) for details, in particular Propositions 5.3 and 5.10.

(i) **risk averse** if $\forall F \in \mathscr{F}$, $\delta_{\mu_F} \succsim F$;

(ii) **risk neutral** if $\forall F \in \mathscr{F}$, $\delta_{\mu_F} \sim F$;

(iii) **risk seeking** if $\forall F \in \mathscr{F}$, $\delta_{\mu_F} \precsim F$.

For every lottery, we assume that there is a value that makes the agent indifferent between taking that lottery and taking this fixed sure value. From this, we obtain the following concepts:

Definition 2. (i) The **certainty equivalent** of F for $\succsim$ is the real number $c(F,\succsim) \in X$ such that $\delta_{c(F,\succsim)} \sim F$.

(ii) The **risk premium** of F for $\succsim$ is the real number $R(F,\succsim) := \mu_F - c(F,\succsim)$.

Exercise 1. *Show that if $\succsim$ be a preference relation on $\mathscr{F}$, and u a strictly increasing expected utility representation of $\succsim$, then $c(F,\succsim)$ is uniquely defined.*

It is intuitive that if an agent is risk averse, then their certainty equivalent is lower then the expected value of a gamble: they would be willing to give up money to avoid risk. Our next result not only shows this, it also relates risk aversion with a structural property of our Bernoulli utility function:

Theorem 1. *Let $\succsim$ be a preference relation on $\mathscr{F}$, and u a strictly increasing expected utility representation of $\succsim$. The following statements are equivalent:*

(i) $\succsim$ is risk averse (risk seeking).

(ii) $c(F,\succsim) \leq (\geq) \mu_F$, $\forall F \in \mathscr{F}$.

(iii) u is concave (convex).

Proof.

(i) $\Longleftrightarrow$ (ii): $\delta_{\mu_F} \succsim F \iff u(\mu_F) = U(\delta_{\mu_F}) \geq U(F) = u(c(F,\succsim))$, where we used monotonicity of u.

(i) $\Longrightarrow$ (iii): $\forall x, x' \in X$, such that $x > x'$, and any $\alpha \in [0,1]$, let F deliver x with probability α and x' with complementary probability. Then $u(\alpha x + (1-\alpha)x') = u(\mu_F) = U(\delta_{\mu_F}) \geq U(F) = \mathbb{E}_F[u] = \alpha u(x) + (1-\alpha)u(x')$.

(i) $\Longleftarrow$ (iii): Take the same F as defined above. Then, $U(\delta_{\mu_F}) = u(\mu_F) \geq \mathbb{E}_F[u] = U(F)$.

The proof of the equivalences for risk seeking preferences is symmetric. $\square$

4. Comparing Risk Attitudes

It is not necessary that a person be consistently risk averse (or risk seeking). For instance, a person may be willing to take a bet for low stakes, but not to take it if the stakes are too high. If so, they are neither risk averse nor risk seeking. Still, we want to compare different people in what regards their risk attitudes:

Definition 3. $\succsim^a$ is said to be more risk averse than $\succsim^b$ if $F \succsim^a \delta_x \implies F \succsim^b \delta_x$, $\forall F \in \mathscr{F}$, $\forall x \in X$.

That is, if whenever person b declines a bet in favour of some sure thing, a more risk averse person a declines too.

We will be able to capture this with an index that summarises their risk attitudes:

Definition 4. For an expected utility representation $u \in C^2$ and $x \in X$, the **Arrow-Pratt coefficient of absolute risk aversion** is given by $r_A(x,u) := -\frac{u''(x)}{u'(x)}$.

What is the coefficient of absolute risk aversion measuring? The *rate* at which marginal utility of wealth changes. Why the rate and not just the curvature? The following exercise will help with that:

Exercise 2. *Let $\succsim$ be a preference relation on $\mathscr{F}$, and $u \in \mathscr{C}^2$ a strictly increasing expected utility representation of $\succsim$.*

1. Show that, $\succsim$ is risk-averse if and only if $r_A(x,u) \geq 0$.

2. If $v \in \mathscr{C}^2$ is another expected utility representation of $\succsim$, then what is the relationship between $r_A(x,v)$ and $r_A(x,u)$?

This coefficient will allow us, later on, to make statements on how risk attitudes change with wealth. Before that, we want to show that this indeed captures how attitudes toward risk of different individuals compare.

Theorem 2. *Let $\succsim^a, \succsim^b$ be two preference relations on $\mathscr{F}$. Let u^a, u^b be strictly increasing expected utility representations of $\succsim^a, \succsim^b$, respectively. The following statements are equivalent:*

(i) $\succsim^a$ is more risk averse than $\succsim^b$.

(ii) $c(F, \succsim^a) \leq c(F, \succsim^b)$, $\forall F \in \mathscr{F}$.

(iii) If $u^b \in \mathscr{C}^0$,[3] then there is a real-valued, strictly increasing, concave function ϕ such that $u^a = \phi \circ u^b$.

(iv) If $u^a, u^b \in \mathscr{C}^2$, then $r_A(x,u^a) \geq r_A(x,u^b)$ for any $x \in X$.

[3]Continuity of u^b is needed to show that (ii) $\implies$ (iii), but not the converse.

Proof.

(i) $\iff$ (ii): $\delta_{c(F,\succsim^a)} \sim^a F \precsim^b \delta_{c(F,\succsim^b)} \iff c(F,\succsim^a) \le c(F,\succsim^b)$.

(ii) $\implies$ (iii): As u^b is strictly increasing, then $u^{b^{-1}}$ is well-defined. As u^a is strictly increasing, then let $\phi := u^a \circ u^{b^{-1}}$. As X is convex and u^b is continuous and strictly increasing, then $u^b(X)$ (the domain of ϕ) is convex.[4] Note that $\phi(u^b(x)) = u^a(u^{b^{-1}}(u^b(x))) = u^a(x)$.
We prove by contrapositive. Suppose that ϕ is not concave. Then, there are $x, x' \in X$, and $\alpha \in (0,1)$, such that $\phi(\alpha u^b(x) + (1-\alpha)u^b(x')) < \alpha\phi(u^b(x)) + (1-\alpha)\phi(u^b(x'))$.
Let F yield x with probability α and x' with complementary probability. Note that $\phi(\alpha u^b(x) + (1-\alpha)u^b(x')) = \phi(\mathbb{E}_F[u^b])$ and $\alpha\phi(u^b(x)) + (1-\alpha)\phi(u^b(x')) = \mathbb{E}_F[\phi \circ u^b]$. Then

$$\begin{aligned} u^a(c(F,\succsim^a)) &= U^a(F) = \mathbb{E}_F[u^a] = \mathbb{E}_F[\phi \circ u^b] \\ &> \phi(\mathbb{E}_F[u^b]) = \phi(U^b(F)) = \phi(u^b(c(F,\succsim^b)) = u^a(c(F,\succsim^b)). \end{aligned}$$

By monotonicity of u^a, we obtain that $c(F,\succsim^a) > c(F,\succsim^b)$.

(ii) $\impliedby$ (iii):

$$\begin{aligned} u^a(c(F,\succsim^a)) &= U^a(F) = \mathbb{E}_F[u^a] = \mathbb{E}_F[\phi \circ u^b] \\ &\le \phi(\mathbb{E}_F[u^b]) = \phi(U^b(F)) = \phi(u^b(c(F,\succsim^b))) = u^a(c(F,\succsim^b)), \end{aligned}$$

which, by strict monotonicity of u^a implies $c(F,\succsim^a) \le c(F,\succsim^b)$.

(iii) $\iff$ (iv): As u^a, u^b are strictly increasing and differentiable, $u^{a\prime}, u^{b\prime} > 0$. As $\phi := u^a \circ u^{b^{-1}}$ and $u^a, u^b \in \mathscr{C}^2$, then $\phi' > 0$ and $\phi \in \mathscr{C}^2$. Moreover, $u^{a\prime\prime}(x) = \phi''(u^b(x))(u^{b\prime}(x))^2 + \phi'(u^b(x))u^{b\prime\prime}(x)$. Then,

$$\begin{aligned} r_A(x,u^b) = -\frac{u^{b\prime\prime}}{u^{b\prime\prime}} &\le -\frac{\phi''(u^b(x))(u^{b\prime}(x))^2 + \phi'(u^b(x))u^{b\prime\prime}(x)}{\phi'(u^b(x))u^{b\prime}(x)} = r_A(x,u^a) \\ &= -\frac{\phi''(u^b(x))u^{b\prime}(x)}{\phi'(u^b(x))} - \frac{u^{b\prime\prime}(x)}{u^{b\prime}(x)} \\ \iff \phi'' &\le 0, \end{aligned}$$

proving the equivalence.

□

5. Risk Attitudes with Changing Wealth

It is popular wisdom that wealthier people are more risk seeking. Or equivalently, risk aversion decreases with wealth. This section will provide us the tool to express these ideas.

Let $F + w$ denote the distribution which arises from adding w to every outcome, i.e. $(F + w)(x) := F(x - w)$. For a preference relation $\succsim$ on $\mathscr{F}$, we will write $\succsim_w$ to denote the preference

[4] This is where continuity of u^b plays a role.

relation of the agent given additional wealth w, i.e. $F \succsim_w G \iff F+w \succsim G+w$. Analogously, we define its expected utility representation $u_w(x) := u(x+w)$, and $U_w(F) = \mathbb{E}_F[u_w]$. Finally, to simplify the statements, let us just assume that $X = \mathbb{R}$.

Definition 5. We say that u exhibits **decreasing/constant/increasing absolute risk aversion** (DARA/CARA/IARA) if $r_A(x,u)$ is decreasing/constant/increasing in x.

Theorem 3. *Let $\succsim$ be a preference relation on $\mathscr{F}$, and u a strictly increasing expected utility representation of $\succsim$. The following statements are equivalent:*

(i) If $u \in \mathscr{C}^2$, u exhibits DARA.

(ii) $\succsim_{w^a}$ is more risk averse than $\succsim_{w^b}$, $\forall w^a \leq w^b$.

(iii) $c(F, \succsim_{w^a}) \leq c(F, \succsim_{w^b})$, $\forall F \in \mathscr{F}$, $\forall w^a \leq w^b$.

(iv) $w^b - w^a \leq c(F+w^b, \succsim) - c(F+w^a, \succsim)$, $\forall F \in \mathscr{F}$, $\forall w^a \leq w^b$.

Proof.

(i) $\iff$ (ii): Follows from Theorem 2(i) $\iff$ (iv).

(ii) $\iff$ (iii): Follows from Theorem 2(i) $\iff$ (ii).

(iii) $\iff$ (iv): This follows from this next lemma.

Lemma 1. *Let $\succsim$ be a preference relation on $\mathscr{F}$, and u a strictly increasing expected utility representation of $\succsim$. Then, $c(F, \succsim_w) = c(F+w, \succsim) - w$.*

Proof.

$$u(c(F, \succsim_w) + w) = u_w(c(F, \succsim_w)) = \mathbb{E}_F[u_w] = \int_X u_w(x) dF(x) = \int_X u(x+w) dF(x)$$
$$= \int_{X+w} u(x) dF(x-w) = \mathbb{E}_{F+w}[u] = u(c(F+w, \succsim)),$$

where $X + w := \{x + w \mid x \in X\}$.[5] □

□

6. Two Functional Forms for Expected Utility

In this section we will see where two extremely common functional forms for Bernoulli utility come from. With this, we gain a better understanding of what we are assuming with adopting that functional form for a given model.

Proposition 1. *For any $u \in \mathscr{C}^2 : r_A(x,u) = \gamma$, $\exists \alpha > 0$, $\beta \in \mathbb{R}$ such that $u(x) = -\alpha sign(\gamma) \exp(-\gamma x) + \beta$ if $\gamma \neq 0$, and $u(x) = \alpha x + \beta$ if otherwise.*

[5]Note that if $X = \mathbb{R}$, then $X + w = \mathbb{R}$.

Proof.

$$r_A(x,u) = -\frac{u''(x)}{u'(x)} = \gamma \iff \int \gamma dx = -\int \frac{u''(x)}{u'(x)}dx \iff \ln u'(x) + k_1 = -\gamma x.$$

If $\gamma \neq 0$, then

$$\ln u'(x) + k_1 = -\gamma x \iff u'(x) = \exp(-\gamma x - k_1) \iff u(x) = -\frac{\exp(-k_1)}{\gamma}\exp(-\gamma x) + k_2,$$

for some $k_1, k_2 \in \mathbb{R}$. If instead $\gamma = 0$, $u''(x) = 0 \implies u(x) = \alpha x + \beta$. □

That is, CARA preferences are completely pinned-down up to positive affine transformations, as with any expected utility representation.

Definition 6. For an expected utility representation $u \in C^2$ and $x \in X$, the **Arrow-Pratt coefficient of relative risk aversion** is given by $r_R(x,u) := -\frac{u''(x)}{u'(x)}x$.

Proposition 2. *For any $u \in \mathscr{C}^2$, such that $r_R(x,u) = \gamma$, $\exists \alpha > 0$, $\beta \in \mathbb{R}$ such that $u(x) = \alpha\frac{x^{1-\gamma}}{1-\gamma} + \beta$, if $\gamma \neq 1$, and $u(x) = \alpha \ln(x) + \beta$ if otherwise.*

Proof.

$$r_R(x,u) = -\frac{u''(x)}{u'(x)}x = \gamma \iff \int \gamma \frac{1}{x} dx = -\int \frac{u''(x)}{u'(x)}dx \iff \ln u'(x) = -\gamma \ln x + k_1$$

$$\iff u'(x) = \exp(k_1)x^{-\gamma} \iff u(x) = \exp(k_1)\frac{x^{1-\gamma}}{1-\gamma} + k_2,$$

for some $k_1, k_2 \in \mathbb{R}$. If $\gamma = 1$, then

$$\frac{u''(x)}{u'(x)} = -\frac{1}{x} \implies \ln u'(x) = -\ln(x) + k_1 \iff u'(x) = \exp(k_1)\frac{1}{x} \iff u(x) = \exp(k_1)\ln(x) + k_2,$$

for some $k_1, k_2 \in \mathbb{R}$. □

An interesting fact about CRRA preferences: it is actually *the only* class of utility functions that, in a Solow model with technological progress at rate g, delivers a balanced growth path, i.e., $\frac{k_{t+1}}{k_t} = \frac{c_{t+1}}{c_t} = 1 + g$.

7. Application

7.1. Buying and Selling Risky Assets

Exercise 3. *Given a gamble $\tilde{x}$ (a real-valued random variable) and a strictly increasing, twice continuously differentiable, $u : \mathbb{R} \to \mathbb{R}$, let the* sale price *$s$ of the gamble is defined by $\mathbb{E}[u(\tilde{x})] = u(s)$. Thus, s is the minimum amount of money the person with the utility function u must be given in order to induce them to give up the gamble $\tilde{x}$.*
If instead they start with no money and no gample, the maximum price they would be willing to pay fo the gamble $\tilde{x}$ is its buy price *b, defined as $\mathbb{E}[u(\tilde{x} - b)] = u(0)$.*

1. *Show that s and b are uniquely defined.*

2. *Show that if u exhibits constant absolute risk aversion, then $b = s$.*

3. *What is the relationship between b and s if u exhibits strictly decreasing absolute risk aversion? Prove your claim.*

4. *Consider another gamble $\tilde{y} = \tilde{x} + c$, where $c \in \mathbb{R}$ is a constant. Let s_y be the sale price of $\tilde{y}$.*

(a) *What is the relationship between s_y and s if u exhibits constant absolute risk aversion? Prove your claim.*

(b) *What is the relationship between s_y and s if u exhibits strictly decreasing absolute risk aversion? Prove your claim.*

7.2. Demand for Risky Assets

Exercise 4. *An agent has $w > 0$ pounds to invest in two assets. The first asset is risk free – for every pound invested, the agent receives $1 + r$ pounds, with $r > 0$. The second asset is risky: for every pound invested, the agent gets a gross return of $\theta \in \Theta$, where θ is a real-valued random variable distributed according to F.*

If the agent invests x pounds in the risky asset, their expected utility is given by $U(x) = \int_\Theta u(x\theta + (1+r)(w-x))dF(\theta)$. Assume that u is concave, strictly increasing, and differentiable.

The agent chooses $x^* \in \arg\max_{x \geq 0} U(x)$. We allow the agent to choose $x > w$, i.e. they can borrow at the risk free rate r.

1. *Show that U is concave.*

2. *Suppose that $\mathbb{E}[\theta] < r$. Solve for the agent's optimal investment decision.*

3. *Suppose that $\mathbb{E}[\theta] = r$. Show that the investment decision found in part 2 is still optimal.*

4. *Suppose that $\mathbb{E}[\theta] > r$. Show that if a solution to the agent's problem exists, then the agent will always invest a strictly positive amount of money on the risky asset.*

7.3. Compounding Risks

Exercise 5. *Let an agent with preferences $\succsim$ over gambles on wealth. Suppose gamble F yields x and $-y$ with equal probability and $x > y$. Let $F^{(n)}$ be a gamble that has n copies of F and $G^{(n)}$ be n copies of a gamble G_n that yields x/n and $-y/n$ with equal probability. That is,*

– *a random variable $w \sim F^{(n)}$ is such that $w = \sum_{i=1}^n w_i$, where each w_i is independently and identically distributed according to F;*

– *a random variable $z \sim G^{(n)}$ is such that $z = \sum_{i=1}^n z_i$, where each z_i is independently and identically distributed according to G_n.*

1. *Prove or disprove:*

(a) If $F \succsim \delta_{\mu_F} \implies F^{(2)} \succsim \delta_{\mu_{F^{(2)}}}$.

(b) If $F \succsim \delta_{\mu_F} \implies G^{(2)} \succsim \delta_{\mu_{G^{(2)}}}$.

2. Assume the agent is risk averse and has a twice continuously differentiable Bernoulli utility u*. Prove or disprove:*

(a) If $F \succsim \delta_0 \implies F^{(2)} \succsim \delta_0$.

(b) If $F \succsim \delta_0 \implies G^{(2)} \succsim \delta_0$.

(c) If n *is large enough, then* $F^{(n)} \succsim \delta_0$.

(d) If n *is large enough, then* $G^{(n)} \succsim \delta_0$.

8. Further Reading

Standard References: Mas-Colell et al. (1995, Chapter 6C), Rubinstein (2018, Chapter 8), Kreps (2012, Chapter 6), Kreps (1988, Chapter 6).

Related questions/topics: While the notion of risk aversion is arguably one of the most ubiquitous concepts in economic theory, empirical and experimental evidence has brought forth some of its limitations. A well-known issue is that people seem to be very risk-averse over small gambles (leading to Rabin's (2000) (in)famous 'calibration theorem'), which poses an issue in interpreting equity premia (search for 'equity premium puzzle'). When discussing expected utility, we mentioned a number of models that try to extend the core elements of expected utility to accommodate for these deviations from the model's predictions.

Let's mention two examples non-expected utility models that deliver risk aversion. One is Yaari's (1987) dual expected utility theory, which is a particular form of rank-dependent utility that can accommodate behaviour like loss-aversion — a concept which captures the intuition that losses loom larger than gains. This model has been recently used to provide insights for auctions and insurance contracts (see Gershkov, Moldovanu, Strack, and Zhang 2022).

Another active research front has been to provide a relation between risk aversion over small stakes and cognitive imprecision (see e.g. Khaw, Li, and Woodford 2022; Frydman and Jin 2022; and Steiner, Netzer, Robson, and Kocourek 2021). This class of models has been making its way into other fields, like finance and macro.

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