ECON0106: **Microeconomics**

Problem Set 9

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Question 1. There are two countries choosing fiscal policies, which determine the tax burden at each income quantile, that is, $s_i : [0,1] \to [0,1]$; we write $S_i := \{s_i : [0,1] \to [0,1]\}$ and, for any $s'_i, s_i \in S_i$, $s'_i \geq s_i$ iff $s'_i(q) \geq s_i(q)$, $\forall q \in [0,1]$.

Each country i observes current market confidence levels, summarised by Θ_i, where we write $\theta'_i \geq \theta_i$ to say that market confidence is higher under θ'_i than under θ_i.

Each country tries to improve its output level, which depends on both countries' tax policies and the own country's market confidence level, as given by $(s_i, s_{-i}, \theta_i) \mapsto u_i(s_i, s_{-i}, \theta_i) \in \mathbb{R}$. Further assume that u_i is continuous in s_i with respect to the product topology, i.e., $u_i(s_i^n, s_{-i}, \theta_i) \to u_i(s_i, s_{-i}, \theta_i)$, $\forall s_i^n : s_i^n(q) \to s_i(q)$ $\forall q \in [0,1]$.

Assume throughout that u_i is quasisupermodular in s_i, i.e., $u_i(s_i, s_{-i}, \theta_i) - u_i(s'_i \wedge s_i, s_{-i}, \theta_i) \geq (>) 0 \implies u_i(s_i \vee s'_i, s_{-i}, \theta_i) - u_i(s'_i, s_{-i}, \theta_i) \geq (>) 0$. (Typo fixed.)

(i) Interpret what quasisupermodularity in s_i of u_i means in this context.

Answer: If, for any two tax policies, it is preferable to lower taxes from $s'_i \vee s_i$ to s_i, then it is also preferable to lower then from s'_i to $s'_i \wedge s_i$.

(ii) Prove that $(S_i, \geq)$ is a complete lattice.

Answer: For any $A \subseteq S_i$, Let $\bar{s}_i(q) = \sup\{s'_i(q), s'_i \in A\}$. Then $\bar{s}_i = \sup_{\geq} A \in S_i$. The argument for $\inf_{\geq} A \in S_i$ is symmetric.

We make two assumptions. (1) Whenever lowering taxes decreases output for country i, then this output loss is lower when market confidence is higher or country j's taxation level is lower. (2) In contrast, whenever lowering taxes increases output for country i, then it would lead to an even greater output increase when market confidence is higher or country j's taxation level is lower.

(iii) Show that, under the above assumptions and regardless of the economic conditions of the countries, there is a (pure strategy) Nash equilibrium in this game.

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Answer: For this and the next subquestion we'll reverse the order on market confidence: Let us define a partial order on market confidence such that $\theta_i' \geq \theta_i \iff \theta_i'$ represents a state of lower market confidence than θ_i.

S_i is bounded. Since $S_i = [0,1]^{[0,1]}$, and $[0,1]$ is compact, by Tychonoff's theorem, S_i is compact. Hence, by Weierstrass extremum theorem, $BR_i(s_{-i},\theta_i) = \arg\max_{s_i \in S_i} u_i(s_i,s_{-i},\theta_i)$ is nonempty. As S_i is a complete lattice and u_i is quasisupermodular in s_i, by Milgrom & Shannon (1994), BR_i is sublattice-valued. Moreover, it is immediate that u_i satisfies strictly increasing differences in $(s_i;s_{-i},\theta_i)$.[1] Hence, by Milgrom & Shannon (1994), BR_i is sublattice-valued, BR_i is strongly increasing in (s_{-i},θ_i). Finally, we get that $BR(s,\theta) := BR_1(s_2,\theta_1) \times BR_2(s_1,\theta_2)$ is increasing in the strong-set order with respect to θ and s. Hence, by Tarski-Zhou's fixed point theorem, there is a pure-strategy Nash equilibrium, i.e., $\exists s \in BR(s,\theta)$, for any $\theta = (\theta_1,\theta_2)$.

(iv) Now suppose that the market confidence of country i increases. What can we say about the equilibrium tax policy of country i? And about country j's tax policy?

Answer: For this subquestion, we keep the reversed order on market confidence such that $\theta_i' \geq \theta_i$ denotes higher market confidence under θ_i than θ_i'.

Let $s^* \in BR(s,\theta)$ and let $\theta_1' > \theta_1$ and $\theta_2' = \theta_2$. Given strictly increasing differences, from Milgrom & Shannon (1994) we obtain that $BR_1(s_2,\theta_1') > BR_1(s_2,\theta_1)$. Then, $\forall s \geq s^*, BR_i(s_j,\theta_i') \geq BR_i(s_j^*,\theta_i') \geq BR_i(s_j^*,\theta_i) = s_i^* \implies BR(s,\theta) \geq s^*$. Hence, restricting the domain of $BR(\cdot,\theta')$ to $S_{\geq s^*} := \{s \in S_1 \times S_2 | s \geq s^*\}$, $BR(\cdot,\theta')$ is a strongly monotone, sublattice-valued correspondence from $S_{\geq s^*}$ to $S_{\geq s^*}$, by Tarski-Zhou's fixed-point theorem, it attains a fixed point on $S_{\geq s^*}$. As $BR(s^*,\theta') \neq s^*$, then $s^{**} \in BR(s^{**},\theta') > s^*$. This implies that the set of equilibrium tax policies (strictly) increase in the weak subset order with respect to θ (with respect to the reversed order): both the highest and lowest equilibrium tax burden of each country decrease in market confidence of either country.

Now consider the alternative assumptions: (1') Whenever lowering taxes decreases output for country i, then this output loss is lower when market confidence is higher or country j's taxation level is higher. (2') In contrast, whenever lowering taxes increases output for country i, then it would lead to an even greater output increase when market confidence is higher or country j's taxation level is higher.

(v) Show that there is a (pure strategy) Nash equilibrium in this game.

Answer: For this subquestion, (a) we keep the reversed order on market confidence; and (b) we reverse the order on s_2. To avoid confusion, we'll define $s_1' \trianglerighteq s_1 \iff s_1' \geq s_1$, $s_2' \trianglerighteq s_2 \iff$

[1] For any $s_i' \geq s_i$, $s_{-i}' \geq s_{-i}$, and $\theta_i' \geq \theta_i$, (1) implies that $u_i(s_i,s_{-i}',\theta_i') - u_i(s_i',s_{-i}',\theta_i') \leq 0 \implies u_i(s_i,s_{-i}',\theta_i') - u_i(s_i',s_{-i}',\theta_i') < u_i(s_i,s_{-i},\theta_i) - u_i(s_i',s_{-i},\theta_i)$, whereas (2) implies that $u_i(s_i,s_{-i}',\theta_i') - u_i(s_i',s_{-i}',\theta_i') \geq 0 \implies u_i(s_i,s_{-i}',\theta_i') - u_i(s_i',s_{-i}',\theta_i') < u_i(s_i,s_{-i},\theta_i) - u_i(s_i',s_{-i},\theta_i)$. Hence, $u_i(s_i,s_{-i}',\theta_i') - u_i(s_i',s_{-i}',\theta_i') < u_i(s_i,s_{-i},\theta_i) - u_i(s_i',s_{-i},\theta_i) \iff u_i(s_i',s_{-i}',\theta_i') - u_i(s_i,s_{-i}',\theta_i') > u_i(s_i',s_{-i},\theta_i) - u_i(s_i,s_{-i},\theta_i)$.

$s'_2 \leq s_2$, and $\theta'_i \trianglerighteq \theta_i \iff \theta'_i$ represents a state of lower market confidence than θ_i.

From the above, we still get that $BR_i(s_{-i},\theta_i)$ is a nonempty-valued, sublattice-valued correspondence (reversing the order won't affect this).

Moreover, it is again straightforward to check that u_i satisfies strictly increasing differences in $(s_i;s_{-i},\theta_i)$ (with respect to the product order based on $\trianglerighteq$).[2] From Milgrom & Shannon (1994), we obtain that BR_i is increasing in strong-set order in (s_{-i},θ_i) (again with respect to $\trianglerighteq$).

Define $BR(s,\theta) := BR_i(s_{-i},\theta_i) \times BR_{-i}(s_i,\theta_{-i})$. Note that $s' \trianglerighteq s \implies s'_i \geq s_i$ and $s'_{-i} \leq s_{-i} \implies$ both $BR_i(\cdot,\theta_i)$ and $BR_{-i}(\cdot,\theta_{-i})$ increase in strong-set order *with respect to* $\trianglerighteq$, hence we obtain that $BR(\cdot,\theta)$ increases in strong-set order *with respect to* $\trianglerighteq$. As $BR_i(\cdot,\theta_i)$ and $BR_{-i}(\cdot,\theta_{-i})$ are nonempty- and sublattice-valued, then so is $BR(\cdot,\theta)$. By Tarski-Zhou's fixed-point theorem, the set of fixed points of $BR(\cdot,\theta)$ is a nonempty, complete lattice.

(vi) Again, suppose that the market confidence of country i increases. What can we say about the equilibrium tax policy of country i? And about country j's tax policy?

Answer: For this subquestion, (a) we keep the reversed order on market confidence; and (b) we reverse the order on s_2. To avoid confusion, we'll define $s'_1 \trianglerighteq s_1 \iff s'_1 \geq s_1$, $s'_2 \trianglerighteq s_2 \iff s'_2 \leq s_2$, and $\theta'_i \trianglerighteq \theta_i \iff \theta'_i$ represents a state of lower market confidence than θ_i.

Let $\bar{b}_i(s_{-i},\theta_i) := \sup_{\geq} BR_i(s_{-i},\theta_i)$ and $\underline{b}_i(s_{-i},\theta_i) := \inf_{\geq} BR_i(s_{-i},\theta_i)$. Note that $s'_1 \geq s_1 \implies \underline{b}_2(s'_1,\theta_2) \leq \underline{b}_2(s_1,\theta_2) \implies \bar{b}_1(\underline{b}_2(s'_1,\theta_2),\theta_1) \geq \bar{b}_1(\underline{b}_2(s_1,\theta_2),\theta_1)$. Hence, $\bar{b}^1(\cdot,\theta) := \bar{b}_1(\cdot,\theta_1) \circ \underline{b}_2(\cdot,\theta_2)$ is, for every fixed θ, an increasing self-map on S_1. The same is true of $\underline{b}^1(\cdot,\theta) := \underline{b}_1(\cdot,\theta_1) \circ \bar{b}_2(\cdot,\theta_2)$.

Suppose $\theta_1 \trianglerighteq \theta'_1$ and $\theta'_2 = \theta_2$. Let s^* and s' be any fixed points of $BR(\cdot,\theta)$ and $BR(\cdot,\theta')$, respectively. From the previous question, we have that $BR_1(s_2,\theta'_1) \geq_{SS} BR_1(s_2,\theta_1) \implies \underline{b}_1(\cdot,\theta'_1) \trianglerighteq \underline{b}_1(\cdot,\theta_1)$ and $\bar{b}_1(\cdot,\theta'_1) \trianglerighteq \bar{b}_1(\cdot,\theta_1) \implies \underline{b}^1(\cdot,\theta') \trianglerighteq \underline{b}^1(\cdot,\theta)$ and $\bar{b}^1(\cdot,\theta') \trianglerighteq \bar{b}^1(\cdot,\theta)$ and, given strictly increasing differences, these inequalities are actually strict.

Then, by a similar argument to that in (iv) — i.e., restricting the domain of $\underline{b}^1(\cdot,\theta'_1)$ and $\bar{b}^1(\cdot,\theta_1)$ to $S_{1 \geq s^*_1}$ and $S_{1 s'_1 \geq}$ respectively, and applying again Tarki's fixed-point theorem — we will find that there are s''_1 and s^{**}_1 such that $s''_1 = \underline{b}^1(s''_1,\theta'_1) \geq s^{**}_1$ and $s^{**}_1 = \bar{b}^1(s^{**}_1,\theta_1) \leq s^*_1$. As this holds for any PSNE s^* and s', and as from (v), the set of PSNE under θ and θ' each constitutes a complete lattice, In particular, this will hold for the largest and smallest fixed points of $BR(\cdot,\theta)$ and $BR(\cdot,\theta')$. Then the set of PSNE increases in the weak set order (with respect to $\trianglerighteq$) going from θ to θ'. In short, both the highest and lowest equilibrium tax burden of each country decrease in own market confidence and increase in the other country's market confidence.

[2] For any $s'_i \trianglerighteq s_i$, $s'_{-i} \trianglerighteq s_{-i}$, and $\theta'_i \trianglerighteq \theta_i$, (1') implies that $u_i(s_i,s'_{-i},\theta'_i) - u_i(s'_i,s'_{-i},\theta'_i) \leq 0 \implies u_i(s_i,s'_{-i},\theta'_i) - u_i(s'_i,s'_{-i},\theta'_i) > u_i(s_i,s'_{-i},\theta'_i) - u_i(s'_i,s'_{-i},\theta'_i)$, whereas (2') implies that $u_i(s_i,s'_{-i},\theta'_i) - u_i(s'_i,s'_{-i},\theta'_i) \geq 0 \implies u_i(s_i,s_{-i},\theta_i) - u_i(s'_i,s_{-i},\theta_i) > u_i(s_i,s'_{-i},\theta'_i) - u_i(s'_i,s'_{-i},\theta'_i)$. Hence, $u_i(s_i,s_{-i},\theta_i) - u_i(s'_i,s_{-i},\theta_i) > u_i(s_i,s'_{-i},\theta'_i) - u_i(s'_i,s'_{-i},\theta'_i) \iff u_i(s'_i,s_{-i},\theta_i) - u_i(s_i,s_{-i},\theta_i) < u_i(s'_i,s'_{-i},\theta'_i) - u_i(s_i,s'_{-i},\theta'_i)$ and u_i satisfies strictly increasing differences in $(s_i;s_{-i},\theta_i)$ (with respect to the product order based on $\trianglerighteq$).

Question 2. Player S, the seller, owns a good that is worth v to player B, the buyer. The value v is drawn from a commonly known distribution F and is private information of the buyer. The players bargain over the price of the good over two periods, with the seller making a take-it-or-leave-it offer (proposing a price) at the start of the first period, that the buyer can accept or reject. The game ends if an offer is accepted or after the two periods, whichever comes first. Both players discount period 2 payoffs with a discount factor of $\delta \in (0,1)$. Assume for simplicity that the buyer always accepts an offer when indifferent.

(i) Characterise all the pure strategy weak perfect Bayesian equilibria of this game when the distribution of v is such that $\mathbb{P}(v = v_H) = 1 - \mathbb{P}(v = v_L) = q \in (0,1)$, where $v_H > v_L > 0$.

(ii) Characterise all the pure strategy weak perfect Bayesian equilibria of this game when v is uniformly distributed on $[\underline{v}, \overline{v}]$.

Question 3. This question serves the purpose of reviewing and practicing the definitions of several of the solution concepts and refinements that we covered throughout the term via a numerical example (getting the definitions right is crucial though).

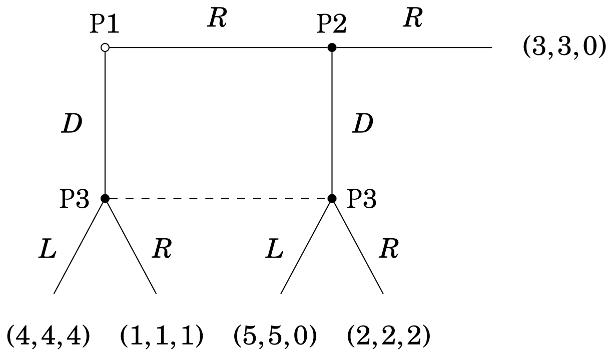

Figure 1:

Refer to the extensive-form game depicted in Figure 1.[3] Show your work; you need to prove your claims.

(i) Describe the normal-form of the game presented above, that is, the set of players, pure strategies for each player, and payoffs.

(ii) Find *all* Nash equilibria (both pure and mixed).

(iii) Find *all* Trembling-Hand Perfect Equilibria.

(iv) Find *all* Subgame-Perfect Nash Equilibria.

(v) Find *all* Weak Perfect Bayesian Equilibria.

(vi) Find *all* Sequential Equilibria.

[3] The payoffs for any given outcome x are given in parentheses $(u_1(x), u_2(x), u_3(x))$, where $u_i(x)$ refers to the payoff of player Pi, $i = 1,2,3$. Note that P1 chooses at the root of the game.