11. Nash Equilibrium

Duarte Gonçalves

University College London

MRes Microconomics

Overview

Before: Setting up the stage, general (but not sharp) predictions.

		Col Player	
		A	B
Row Player	A	1, 0	0, 1
	B	0, 1	1, 0

Please predict choice frequencies.

Now indicate strategies surviving IESDS? And rationalisable strategies?

Now: Sharpening prediction, bridging the disconnect.

Goal: Nash equilibrium, GT's gold standard.

Used *everywhere*. (Not just economics or even social sciences.)

Overview

1. Motivation
2. Nash Equilibrium
3. Examples
4. Normal-Form Refinements and Generalisations of Nash Equilibrium
5. More

Overview

1. Motivation

2. Nash Equilibrium
   - Definition and Interpretations
   - Existence of a Nash Equilibrium
   - Relation to Dominance
   - Characterising Equilibria
   - Interpreting MSNE
   - Robustness

3. Examples

4. Normal-Form Refinements and Generalisations of Nash Equilibrium

5. More

Nash Equilibrium

Definition

$s \in S$ is a **pure strategy Nash equilibrium** iff $\forall i$, $u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$ $\forall s'_i \in S_i$ (s_i is BR to s_{-i}).

$\boldsymbol{\sigma} \in \Sigma$ is a **Nash equilibrium** if $\forall i$, $u_i(\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_{-i}) \geq u_i(\boldsymbol{\sigma}'_i, \boldsymbol{\sigma}_{-i})$ $\forall \boldsymbol{\sigma}'_i \in \Sigma_i$ ($\boldsymbol{\sigma}_i$ is BR to $\boldsymbol{\sigma}_{-i}$).

Note the difference: equilibrium, equilibrium payoff, equilibrium outcome.

Pure strategy $\equiv$ degenerate mixed strategies.

Non-degenerate mixed strategies $\not\equiv$ totally mixed strategies.

Nash Equilibrium

Definition

Player i's best-response correspondence is given by $b_i : \Sigma_{-i} \rightrightarrows \Sigma_i$ s.t. $b_i(\boldsymbol{\sigma}_{-i}) := \arg\max_{\boldsymbol{\sigma}_i \in \Sigma_i} u_i(\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_{-i})$.

$b : \Sigma \rightrightarrows \Sigma$ s.t. $b(\boldsymbol{\sigma}) := \times_{i \in I} b_i(\boldsymbol{\sigma}_{-i})$ denotes players' best-response correspondence.

Remark

$\boldsymbol{\sigma}$ is a Nash equilibrium iff $\boldsymbol{\sigma} \in b(\boldsymbol{\sigma})$.

Nash Equilibrium: Interpretation

(1) **Resulting from introspection**.

Everyone is BR to everyone else. If not, they'd prefer to do something else.

NE is rationalisable. All rationalisable strategy profiles are NE? No: NE $\subset$ Rationalisable.

Epistemic Foundations: Aumann & Brandenburger (1995 Ecta)

Requires conjecture about what opponents are playing and conjecture being right.

We all do the best given what the other is doing; no one has an incentive to deviate.

Example: You can pick 0 or 1. You win if you get the closest to 3/2 of the class average. Write down a number.

What is your prediction?

Generally: What do to when there are multiple equilibria? How does one decide which one is to be played?

Nash Equilibrium: Interpretation

(1) **Resulting from introspection**.

(2) **An outcome of learning, a steady state of a long-run adjustment process.**

Can help in selection of an equilibrium.

Fudenberg & Levine (1998); see also Fudenberg & Levine (2016) and Fudenberg (2022) for surveys.

Learning and dynamic adjustment: in next year's theory topics course!

Growing literature on estimating equilibria in games; what about dynamic adjust toward equilibrium? (workshop on GT & metrics sponsored by cemmap)

Existence of a Nash Equilibrium

Who cares about existence? **You do!**

- You want your model to make predictions, to explain something about the world.
- Equilibrium here means that everyone's actions are consistent / your model makes sense / the system is consistent.
- Existence of a solution simply means the model has something to say about a given situation (under a set of assumptions).
- Write down assumptions formally, but there is no equilibrium; your model is unable to make predictions!
- Not a good model of the agents' behaviour.

Existence results ensure that, under a given set of assumptions, your model *works* (whether it makes good predictions or not it is another matter)

Existence of a Nash Equilibrium

Kakutani's Fixed-Point Theorem

Let $X \subset \mathbb{R}^n$ be nonempty, compact, and convex. If $F : X \rightrightarrows X$ is nonempty-valued, compact-valued, convex-valued, and uhc, then $\exists x \in X : x \in F(x)$, i.e., there is a fixed point of F.

Existence of a Nash Equilibrium

Theorem

Let $\Gamma = \langle I, S, u \rangle$ be a normal-form game s.t. $|I| < \infty$, and, $\forall i \in I$, S_i is a nonempty, compact, and convex subset of $\mathbb{R}^n$. If $u_i : S \to \mathbb{R}$ is continuous in S and quasiconcave in s_i, then, there is a PSNE.

Proof

Let $b : S \rightrightarrows S$ be s.t. $b(s) = \times_{i \in I} b_i(s_{-i})$, where $b_i(s_{-i}) := \arg\max_{s_i \in S_i} u_i(s_i, s_{-i})$.

(i) b is nonempty-, compact-valued, and UHC.

u_i continuous, S_i is compact $\implies$ b_i nonempty-value, compact-valued and UHC $\forall i \in I$ (by Berge's maximum theorem).

Immediately $\implies$ b nonempty-value, compact-valued, and UHC (finite Cartesian product of nonempty and compact sets is nonempty and compact wrt to product metric; for UHC use definition to verify).

(ii) b is convex-valued.

u_i is quasiconcave in s_i $\implies$ $b_i(s_{-i})$ is convex $\forall s_{-i} \in S_{-i}, \forall i \in I$ $\implies$ b convex-valued.

By Kakutani's fixed-point theorem, $\exists s \in b(s)$. $\square$

Existence of a Nash Equilibrium

Theorem

Let $\Gamma = \langle I, S, u \rangle$ be a normal-form game s.t. $|I| < \infty$, and, $\forall i \in I$, S_i is a nonempty, compact, and convex subset of $\mathbb{R}^n$. If $u_i : S \to \mathbb{R}$ is continuous in S and quasiconcave in s_i, then, there is a PSNE.

What about $\exists$ in finite games? A special case...

Corollary

Let $\Gamma = \langle I, S, u \rangle$ be a normal-form game s.t. $|I|, |S| < \infty$. Then, there is a NE, possibly in mixed-strategies.

Proof

- Game in mixed-strategies as a different game, $\tilde{\Gamma} = \langle I, \Sigma, \tilde{u} \rangle$, with $\tilde{u}_i(\boldsymbol{\sigma}) = \mathbb{E}_{\boldsymbol{\sigma}}[u_i]$.
- Σ_i as a nonempty, compact, and convex subset of $[0, 1]^{|S_i|}$
- $\tilde{u}_i$ continuous in $\boldsymbol{\sigma}$ and linear (hence quasiconcave) in $\boldsymbol{\sigma}_i$.
- Conditions of theorem met, hence $\exists$ PSNE $\boldsymbol{\sigma}$ of $\tilde{\Gamma}$, which is NE (possibly mixed) of Γ. $\square$

Existence of a Nash Equilibrium

Nash provided a different proof, based on Brouwer's fixed point theorem:

Brouwer's Fixed-Point Theorem

Let X be a nonempty, compact, and convex subset of $\mathbb{R}^n$. If $f : X \to X$ is continuous, then f admits a fixed-point $x = f(x)$.

Instructive proof by Geneakoplos.

Theorem

Let $\Gamma = \langle I, S, u \rangle$ be a normal-form game s.t. $|I| < \infty$, and, $\forall i \in I$, S_i is a nonempty, compact, and convex subset of $\mathbb{R}^n$. If $u_i : S \to \mathbb{R}$ is continuous in S and concave in s_i, then there is a PSNE.

Existence of a Nash Equilibrium

Proof

- Let $\phi_i : S \to S_i$ be s.t. $\phi_i(s) := \arg\max_{s'_i \in S_i} u_i(s'_i, s_{-i}) - ||s_i - s'_i||^2$.
- $u_i(s'_i, s_{-i})$ is concave in s'_i and $-||s_i - s'_i||^2$ is strictly concave
  $\implies u_i(s'_i, s_{-i}) - ||s_i - s'_i||^2$ strictly concave and continuous in s'_i.
- S_i convex and compact, $u_i(s'_i, s_{-i}) - ||s_i - s'_i||^2$ strictly concave and continuous in s'_i
  $\implies \phi_i$ is well-defined (singleton maximiser).
- Berge's maximum theorem $\implies \phi_i$ UHC + singleton-valued $\implies \phi_i$ continuous.
- Let $\phi(s) := (\phi_i(s))_{i \in I}$. $\phi : S \rightrightarrows S$ continuous, $S \subset \mathbb{R}^n$ convex, compact $\implies \phi$ has fixed point $s = \phi(s)$ (by Brouwer's fixed-point theorem).
- WTS $s_i \in \arg\max_{s_i \in S_i} u_i(s_i, s_{-i})$.

Existence of a Nash Equilibrium

Proof

- $\phi_i(s) := \arg\max_{s_i' \in S_i} u_i(s_i', s_{-i}) - ||s_i - s_i'||^2$. $\phi(s) := (\phi_i(s))_{i \in I}$. $\exists s \in S : s = \phi(s)$.
- WTS $s_i \in \arg\max_{s_i \in S_i} u_i(s_i, s_{-i})$.
  - Suppose not, i.e., $\exists s_i' \in S_i : u_i(s_i', s_{-i}) > u_i(s)$ for some i.
  - u_i concave $\implies u_i(\alpha s_i' + (1-\alpha)s_i, s_{-i}) - u_i(s) \geq \alpha(u_i(s_i', s_{-i}) - u_i(s))\ \forall \alpha \in (0,1)$.
  - NB: $||s_i - (\alpha s_i' + (1-\alpha)s_i)||^2 = \alpha^2 ||s_i - s_i'||^2$.
  - As $\phi_i(s) = s_i \implies \forall \alpha \in (0,1)$

$$\begin{aligned}
0 \geq & u_i((\alpha s_i' + (1-\alpha)s_i), s_{-i}) - ||s_i - (\alpha s_i' + (1-\alpha)s_i)||^2 \\
& - \left(\max_{s_i'' \in S_i} u_i(s_i'', s_{-i}) - ||s_i - s_i''||^2 \right) \\
= & u_i((\alpha s_i' + (1-\alpha)s_i), s_{-i}) - ||s_i - (\alpha s_i' + (1-\alpha)s_i)||^2 - u_i(s) \\
\geq & \alpha(u_i(s_i', s_{-i}) - u_i(s)) - \alpha^2 ||s_i - s_i'||^2
\end{aligned}$$

  - Choosing α s.t. $\alpha < (u_i(s_i', s_{-i}) - u_i(s))/||s_i - s_i'||^2$ delivers
    $0 \geq \alpha(u_i(s_i', s_{-i}) - u_i(s)) - \alpha^2 ||s_i - s_i'||^2 > \alpha^2 - \alpha^2 = 0$. a contradiction. $\square$

Symmetric Nash Equilibria

Definition

Let $\Gamma = \langle I, S, u \rangle$ be a normal-form game. Γ is **symmetric** iff $\forall i, j \in I$, $S_j = S_i$, and $u_i(s_i, s_{-i}) = u_j(s_j, s_{-j})$ for $s_i = s_j$ and $s_{-i} = s_{-j}$.

Proposition

Let $\Gamma = \langle I, S, u \rangle$ be a symmetric normal-form game s.t. $|I| < \infty$, and, $\forall i \in I$, S_i is a nonempty, compact, and convex subset of $\mathbb{R}^n$. If $u_i : S \to \mathbb{R}$ is continuous in S and quasiconcave in s_i, then there is a PSNE s s.t. $s_i = s_j \in S_i \forall i, j \in I$.

Proof

Define $b : S_i \rightrightarrows S_i$ s.t. $\tilde{b}(s_i) = b_i(s_{-i})$ for $s_{-i} = (s_i)_{j \in -i}$; b_i is player i's best-response correspondence.

$\tilde{b}$ nonempty-, compact-, and convex-valued, and UHC
$\implies$ Kakutani's fixed-point theorem applies. $\square$

Existence of a Nash Equilibrium

If strategy space not finite, can we do better? Continuity is enough:

Theorem

Let $\Gamma = \langle I, S, u \rangle$ be a normal-form game s.t. $|I| < \infty$, and, $\forall i \in I$, S_i is a nonempty, compact, and convex subset of $\mathbb{R}^n$. If $u_i : S \to \mathbb{R}$ is continuous, then there is a NE, possibly in mixed strategies.

Relies on the following generalisation of Kakutani's fixed-point theorem:

Theorem

Let X be a nonempty, compact, convex subset of a locally convex Hausdorff (e.g., vector) space and that $f : X \rightrightarrows X$ is nonempty- and convex-value correspondence with a closed graph. Then $\exists x \in X : x \in f(x)$.

Does 2PA have continuous payoffs? What to do then? See Reny (1999 Ecta) "On the Existence of Pure and Mixed Strategy Nash Equilibria in Discontinuous Games"

Relation to Dominance

(i) If game is dominance solvable, then dominance solution is the unique NE of the game.

(ii) Any Nash equilibrium strategy must be rationalisable (and thus survive IESDS).

(iii) Any pure strategy in the support of a Nash equilibrium is also rationalisable.

(iv) However... weakly dominated strategies *can* be played with positive probability at a Nash equilibrium.

Relation to Dominance

		Col Player A	Col Player B
Row Player	A	1,1	0,0
	B	0,0	0,0

NE: (A,A) and (B,B)

Characterising Equilibria

		Col Player	
		A	B
Row Player	A	1,1	2,3
	B	3,2	0,0

PSNE: (A,B) and (B,A)

Characterising Equilibria

Remark

$\boldsymbol{\sigma}$ is a Nash equilibrium if and only if $\forall i \in I$, (i) $u_i(s_i, \boldsymbol{\sigma}_{-i}) \geq u_i(s_i', \boldsymbol{\sigma}_{-i})$ $\forall s_i \in \text{supp}(\boldsymbol{\sigma}_i), s_i' \in S_i$, and (ii) $u_i(s_i, \boldsymbol{\sigma}_{-i}) = u_i(s_i', \boldsymbol{\sigma}_{-i})$ $\forall s_i, s_i' \in \text{supp}(\boldsymbol{\sigma}_i)$.

Best response condition: any pure strategy in the support must be best response to $\boldsymbol{\sigma}_{-i}$, $u_i(s_i, \boldsymbol{\sigma}_{-i}) \geq u_i(s_i', \boldsymbol{\sigma}_{-i})$ for any $s_i \in \text{supp}(\boldsymbol{\sigma}_i)$ and $s_i' \in S_i$.

MSNE indifference condition: must get same payoff $u_i(s_i, \boldsymbol{\sigma}_{-i}) = u_i(s_i', \boldsymbol{\sigma}_{-i})$ for any pure strategy in the support, $s_i, s_i' \in \text{supp}(\boldsymbol{\sigma}_i)$.

Characterising Equilibria

		Col Player	
		A	B
Row Player	A	1,1	2,3
	B	3,2	0,0

PSNE: (A,B) and (B,A)

Given σ_C,

$$u_R(A, \sigma_C) \geq u_R(B, \sigma_C) \implies \sigma_C(A)1 + (1 - \sigma_C(A))2 \geq \sigma_C(A)3 + (1 - \sigma_C(A))0$$

$$\sigma_C(A) \leq \frac{1}{2}.$$

NE: (A,B), (B,A), and $(\sigma_R, \sigma_C) : \sigma_R(A) = \sigma_C(A) = 1/2$.

Issues with MSNE

Players don't willingly randomise; they are *indifferent*. How to interpret MSNE?

As an equilibrium in beliefs (Aumann & Brandenburger 1995 Ecta).

As outcome of long-run learning/adjustment process.

As stochastic choice: players look like they are randomising, but could be random utility players ('purification' e.g. Harsanyi 1973 IJGT – more later), unobserved information acquisition (Gonçalves 2024 WP), etc.

MSNE *can be* rationalised as the limit outcome of one such situation.

Robustness

Will limit of equilibria be an equilibrium of the limit game? (is set of NE UHC?)

Robustness

Proposition

Let $S := \times_{i \in I} S_i$ be such that S_i is nonempty, compact, and convex subset of $\mathbb{R}^{n_i}$, $T \subseteq \mathbb{R}^m$, $u_i : S \times T \to \mathbb{R}$. Let $S^{NE}(t) := \left\{ s \in S \mid s_i \in \arg\max_{s'_i \in S_i} u_i(s'_i, s_{-i}, t) \right\}$ be set of NE of the game $\Gamma_t := \langle I, S, u^t \rangle$, where $u^t := (u_i(\cdot, t))_{i \in I}$.

If (i) u_i is continuous in (s, t), and (ii) $S^{NE}(t')$ is nonempty for any t' in a neighborhood of t, then S^{NE} is UHC at t.

Proof

- Take $(s^n, t^n) \to (s, t)$, where $s^n \in S^{NE}(t^n)$ for all n.
- Then $u_i(s_i^n, s_{-i}^n, t^n) = \max_{s'_i \in S_i} u_i(s'_i, s_{-i}^n, t^n)$.
- By Berge's maximum theorem, $\max_{s'_i \in S_i} u_i(s'_i, s_{-i}, t)$ is continuous in (s_{-i}, t).
- Then

$$u_i(s_i, s_{-i}, t) = \lim_{n \to \infty} u_i(s_i^n, s_{-i}^n, t^n) = \lim_{n \to \infty} \max_{s'_i \in S_i} u_i(s'_i, s_{-i}^n, t^n) = \max_{s'_i \in S_i} u_i(s'_i, s_{-i}, t).$$

- Hence, $s \in S^{NE}(t)$ and S^{NE} is uhc (and compact-valued) at t. $\square$

Robustness

Will limit of equilibria be an equilibrium of the limit game? (is set of NE UHC?) **Yes**

Will equilibrium of a game be similar to equilibria of nearby games? (is set of NE LHC?)

		Col Player	
		A	B
Row Player	A	1,1	0,0
	B	0,0	0,0

(B,B) is NE.

Look at nearby games:

		Col Player	
		A	B
Row Player	A	1,1	$1/n$,$1/n$
	B	$1/n$,$1/n$	$-1/n$,$-1/n$

For any $n > 1$, unique equilibrium is (A,A). In the limit, $n \to \infty$, (B,B) also NE.

$\implies$ Set of NE not LHC.

Issue: not all NE are robust; small mistakes may 'kill fragile equilibria'. More later.

Overview

1. Motivation
2. Nash Equilibrium
3. Examples
   - Common Value All-Pay Auction
   - Model of Sales
4. Normal-Form Refinements and Generalisations of Nash Equilibrium
5. More

Common Value All-Pay Auction

I bidders, all value object at $v > 0$. Bids $s_i \geq 0$.

Payoffs: always pay bid; win if bid highest; ties broken uniformly at random.

$$u_i(s_i, s_{-i}) = 1\{s_i = \max_j s_j\} \cdot \frac{1}{|\{j \in I \mid s_j = \max_\ell s_\ell\}|} v - s_i$$

Claim 1: No one bids above v: strictly dominated.

Claim 2: No PSNE in this game.

Suppose s is PSNE.

(a) If $\max_j s_j < v \implies \forall i \notin \arg\max_j s_j \implies i$ wants to deviate to $s_i' = \max_j s_j + \varepsilon$.

(b) $i = \arg\max_j s_j$, then i wants to deviate to $s_i' = s_i - \varepsilon$ for small enough $\varepsilon > 0$.

(c) If $\max_j s_j = v$, $i, \ell \in \arg\max_j s_j$ with $i \neq \ell$, then i wants to deviate to $s_i' = 0$.

Common Value All-Pay Auction

Is there NE? Let's try to construct a symmetric MSNE $\sigma_i = \sigma_j$ (CDF):

- NB: $\sigma_i \in \Delta([0, v])$; (why?)
  As $\forall s_i > v$, s_i is strictly dominated by $s_i' = 0 \implies$ bids $> v$ occur wp 0 at any NE.
  $\implies \sigma_j(v) = 1$.
- *Assume* has no atoms. (We'll verify later.)
- Player i needs to be indifferent over all bids in support:
  if $s_i \in \text{supp}(\sigma_i)$, then

$$\overline{u} = u_i(s_i, \sigma_{-i}) = \mathbb{P}(s_i > \max_{j \neq i} s_j)v - s_i = \mathbb{P}(s_i > s_j)^{I-1} v - s_i = \sigma_j(s_i)^{I-1} v - s_i$$

$$\iff \sigma_j(s_i) = \left(\frac{\overline{u} + s_i}{v}\right)^{\frac{1}{I-1}}$$

- As $\sigma_j(v) = 1 \iff \overline{u} = 0$.
- No atoms: $\sigma_j(s_j) = 0 \implies s_j = 0$.
- By construction, $\forall s_i \in [0, v]$, $u_i(s_i, \sigma_{-i}) = \overline{u} = 0$. (Indifference).
- Conclusion: Competition left bidders with 0 expected surplus from the auction.

Model of Sales (Varian 1980 AER)

Two firms sell a good to a unit mass of consumers with reservation price of £1 (WTP).

Firms set prices simultaneously.

Each firm has loyal consumers (insofar as price doesn't exceed reservation price):

1/4 of consumers always buy from firm 1 and 1/4 from firm 2.

Remaining 1/2 are "active shoppers" and buy from the lower-price firm.

Construct a PSNE. Ideas?

Claim: There is no PSNE in this game.

Any $p_i > 1$ or $p_i = 0$ is strictly dominated by $p_i' = 1$.

For $p : p_i, p_j : p_i = p_j > 0$, strictly profitable for i to deviate to $p_i' = p_j - \varepsilon$ for small ε.

For $p : p_i, p_j : 1 = p_i > p_j$, strictly profitable for j to deviate to $p_j' = 1 - (1 - p_j)/2$.

For $p : p_i, p_j : 1 > p_i > p_j > 1/3$, strictly profitable for i to deviate to $p_i' = p_j - \varepsilon$ for small ε and get $\frac{3}{4}(p_j - \varepsilon) > \frac{1}{4} \geq \frac{1}{4}p_i$.

For $p : p_i, p_j : 1 > p_i > p_j$ and $1/3 \geq p_i$, strictly profitable for i to deviate to $p_i' = 1$ and get $\frac{1}{4} \geq \frac{1}{4}p_i$.

Model of Sales (Varian 1980 AER)

Two firms sell a good to a unit mass of consumers with reservation price of £1 (WTP).

Firms set prices simultaneously.

Each firm has loyal consumers (insofar as price doesn't exceed reservation price):

1/4 of consumers always buy from firm 1 and 1/4 from firm 2.

Remaining 1/2 are "active shoppers" and buy from the lower-price firm.

Claim: There is no PSNE in this game.

Construct a MSNE.

(i) Assume no atoms, symmetric NE. Support $[\underline{p}, \overline{p}] \subseteq (0, 1]$.

(ii) Indifference: $\frac{1}{4}p_i + \frac{1}{2}(1 - \sigma_j(p_i))p_i = \overline{\pi} \iff \sigma_j(p_i) = \frac{3}{2} - 2\frac{\overline{\pi}}{p_i}$.

(iii) No firm prices above £1 (strictly dominated)

$1 = \sigma_j(1) = 3/2 - 2\overline{\pi} \iff \overline{\pi} = \frac{1}{4}$.

(iv) No firm prices below £1/3: $\forall p_i > 0$, $0 = \sigma_j(p_i) = \frac{3}{2} - \frac{1}{2p_i} \iff p_i = \frac{1}{3}$.

(Note this is same as extracting all surplus out of the loyal consumers.)

(v) Pricing below $1/3$ yields $\frac{3}{4}p_i < \frac{1}{4}$.

(i.e., in expectation worse than pricing at $p_i \in [1/3, 1]$.)

Overview

1. Motivation

2. Nash Equilibrium

3. Examples

4. Normal-Form Refinements and Generalisations of Nash Equilibrium
   - Trembling-Hand Perfection
   - Correlated Equilibrium

5. More

A Silly Story

A group of friends is going to the movies and they are deciding which movie to watch.

The options are *Heretic* and *Megalopolis* and they'll watch the most voted movie.

They have all watched *Megalopolis* (hate-watching) and so no one really wants to watch it again (once is more than enough). Everyone prefers to watch Heretic.

But there is a NE in which everyone votes to watch *Megalopolis* again!

This is very silly and we want to rule out silly predictions in our model.

Trembling-Hand Perfection

Idea: non-zero probabilities on each pure strategy capture the notion of unavoidable mistakes.

Define $\Delta_\varepsilon(S_i) := \{\sigma_i \in \Sigma_i \mid \sigma_i(s_i) \geq \varepsilon(s_i), \forall s_i \in S_i\}$ for $\varepsilon : \cup_{i \in I} S_i \to (0,1)$.

$\Delta_\varepsilon(S_i)$: a restricted strategy space for player i, with fully mixed strategies.

Definition

An ε-constrained Nash equilibrium of game $\Gamma = \langle I, S, u \rangle$ is a pure strategy Nash equilibrium of the perturbed game $\langle I, \times_i \Delta_\varepsilon(S_i), u \rangle$.

Remark

For any ε, $\Delta_\varepsilon(S_i)$ is compact. Hence, insofar as $\Delta_\varepsilon(S_i)$ is nonempty for all i, there is an ε-constrained NE.

Trembling-Hand Perfection

Definition

A NE $\boldsymbol{\sigma}$ of game Γ is trembling-hand perfect if $\exists(\boldsymbol{\varepsilon}^n)_n$ s.t. $\boldsymbol{\varepsilon}^n : \cup_{i\in I} S_i \to (\mathbf{0},\mathbf{1})$ with $\boldsymbol{\varepsilon}^n(s_i) \to \mathbf{0}\ \forall s_i \in S_i$, and an associated sequence of $\boldsymbol{\varepsilon}^n$-constrained NE $\boldsymbol{\sigma}^n : \boldsymbol{\sigma}^n \to \boldsymbol{\sigma}$.

Theorem

Every finite game has a THPE.

Proof

Let $\tilde{\boldsymbol{\varepsilon}}^n := \max_{s_i \in \cup_{i\in I}} \boldsymbol{\varepsilon}^n(s_i)$.

By convergence of $\tilde{\boldsymbol{\varepsilon}}^n \to \mathbf{0}$, $\exists N : \forall n > N$, $\Delta_{\boldsymbol{\varepsilon}^n}(S_i) \neq \emptyset \forall i \in I$.

Then, $\forall n > N\ \exists \boldsymbol{\varepsilon}^n$-constrained equilibrium.

Take any sequence of $\boldsymbol{\varepsilon}^n$-constrained equilibrium for $n > N$; as it lives in a compact set, it admits a convergent subsequence.

As $u_i : \Sigma \to \mathbb{R}$ is continuous $\forall i$, subsequence converges to a NE of original game.

Trembling-Hand Perfection

Proposition

A NE σ of game Γ is THPE iff $\exists$ sequence of fully mixed strategy profiles $\sigma^n \to \sigma : \forall i$ and n, σ_i is a best response to σ^n_{-i}.

Proof: $\implies$

If σ is THPE $\implies \exists \varepsilon^n : \sigma$ is limit of ε^n-constrained NE σ^n.

WTS $\exists$ subsequence σ^m of σ^n s.t. σ_i is a best response to σ^m_{-i}.

- Suppose not. Then $\exists s_i \in \operatorname{supp}(\sigma_i)$ and some further subsequence σ^k s.t. $\forall k$, $u_i(s_i, \sigma^k_{-i}) < u_i(s^k_i, \sigma^k_{-i})$ for some $s^k_i \in S_i$.
- Let $\tilde{\varepsilon}^n := \max_{s_i \in \cup_{i \in I}} \varepsilon^n(s_i)$. We must have $\sigma^k_i(s_i) \leq \tilde{\varepsilon}^k$. Supposing otherwise yields:

$$u_i(\sigma^k_i, \sigma^k_{-i}) = \sigma^k_i(s_i) u_i(s_i, \sigma^k_{-i}) + \sigma^k_i(s^k_i) u_i(s^k_i, \sigma^k_{-i}) + \sum_{s'_i \neq s_i, s^k_i} \sigma_i(s'_i) u_i(s'_i, \sigma^k_{-i})$$

$$< \tilde{\varepsilon}^k u_i(s_i, \sigma^k_{-i}) + (\sigma^k_i(s^k_i) + \sigma^k_i(s_i) - \varepsilon^k) u_i(s^k_i, \sigma^k_{-i}) + \sum_{s'_i \neq s_i, s^k_i} \sigma_i(s'_i) u_i(s'_i, \sigma^k_{-i}).$$

- But then $\sigma^k_i(s_i) \to 0$, which contradicts the fact that $\sigma^k_i(s_i) \to \sigma_i(s_i) > 0$.

Trembling-Hand Perfection

Proof: $\Longleftarrow$

- Let $\varepsilon^n(s_i) = \sigma_i^n(s_i)$ if $\sigma_i(s_i) = 0$ and $\varepsilon^n(s_i) = \sigma_i(s_i)/n$ if $\sigma_i(s_i) > 0$.
- As σ_i is BR to $\sigma_{-i}^n \implies \forall s_i \in \text{supp}(\sigma_i) =: A$ is BR to σ_{-i}^n.
- Let $\overline{u}^n := u_i(s_i, \sigma_{-i}^n) \geq u_i(s_i', \sigma_{-i}^n)\ \forall s_i' \in S_i$. Let $A := \text{supp}(\sigma_i)$ and $B := S_i \setminus A$.
- Then, $\forall \sigma_i' \in \Delta_{\varepsilon^n}(S_i)$,

$$\begin{aligned}
u_i(\sigma_i', \sigma_{-i}^n) &= \sum_{s_i \in A} \sigma_i'(s_i) u_i(s_i, \sigma_{-i}^n) + \sum_{s_i \in B} \sigma_i'(s_i) u_i(s_i, \sigma_{-i}^n) \\
&= \sum_{s_i \in A} \sigma_i'(s_i) \overline{u}^n + \sum_{s_i \in B} \left(\sigma_i'(s_i) - \sigma_i^n(s_i)\right) u_i(s_i, \sigma_{-i}^n) + \sum_{s_i \in B} \sigma_i^n(s_i) u_i(s_i, \sigma_{-i}^n) \\
&\leq \sum_{s_i \in A} \sigma_i'(s_i) \overline{u}^n + \sum_{s_i \in B} \left(\sigma_i'(s_i) - \sigma_i^n(s_i)\right) \overline{u}^n + \sum_{s_i \in B} \sigma_i^n(s_i) u_i(s_i, \sigma_{-i}^n) \\
&= \sum_{s_i \in A} \sigma_i^n(s_i) \overline{u}^n + \sum_{s_i \in B} \sigma_i^n(s_i) u_i(s_i, \sigma_{-i}^n) \\
&= u_i(\sigma_i^n, \sigma_{-i}^n)
\end{aligned}$$

and therefore σ_i^n is ε^n-constrained BR to σ_{-i}^n. (Feasible $\forall n > 2|S|$.) $\square$

Trembling-Hand Perfection

Proposition

(i) If σ is fully mixed NE, then it is THPE.
- Take itself as a sequence of fully mixed.

(ii) If σ is THPE, then no strategy assigns positive probability to weakly dominated strategies.
- Suppose it does, then it cannot be a best response to any fully mixed perturbation of the opponents' strategy.

(iii) In 2-player games, NE assigns prob zero to weakly dominated strategies iff it is THPE. • Exercise.

(iv) Any NE σ such that $u_i(\sigma_i, \sigma_{-i}) > u_i(\sigma_i', \sigma_{-i})$ for any $\sigma_i \neq \sigma_i'$ is THPE.
- This is **strict NE**. NB: any strict NE must be in pure strategies (degenerate mixed strategies).

  Continuity implies strict NE is strict BR to *any* small enough trembles of the opponents' strategy (including fully mixed).

Other notions: strong or strict equilibrium

Correlated Equilibrium

And Now for Something Completely Different...

Definition

A probability distribution $p \in \Delta(S)$ is a **correlated equilibrium** of a normal-form game $\Gamma = \langle I, S, u \rangle$ if $\forall i$ and $\forall s_i : p(s_i) > 0$

$$\sum_{s_{-i} \in S_{-i}} p(s_{-i}|s_i) u_i(s_i, s_{-i}) \geq \sum_{s_{-i} \in S_{-i}} p(s_{-i}|s_i) u_i(s_i', s_{-i}), \quad \forall s_i' \in S_i.$$

p as correlated recommendations to the agents s.t. everyone wants to follow a the recommendations.

Proposition

Every Nash equilibrium is a correlated equilibrium

Corollary

CE exists in finite games

Correlated Equilibrium

Definition

A probability distribution $p \in \Delta(S)$ is a **correlated equilibrium** of a normal-form game $\Gamma = \langle I, S, u \rangle$ if $\forall i$ and $\forall s_i : p(s_i) > 0$

$$\sum_{s_{-i} \in S_{-i}} p(s_{-i}|s_i) u_i(s_i, s_{-i}) \geq \sum_{s_{-i} \in S_{-i}} p(s_{-i}|s_i) u_i(s_i', s_{-i}), \quad \forall s_i' \in S_i.$$

Usefulness: Easily computable. Mediator trying to get an outcome to emerge. (Example: advertising a mega party.)

Convex hull of set of NE is subset of set of CE (also convex).

$\implies$ Can attain any convex combination of NE payoffs. That's it?

Correlated Equilibrium: An Example

Coordination Game

		Col Player A	Col Player B
Row Player	A	5,1	0,0
	B	4,4	1,5

- NE? (A,A), (B,B), and (1/2 A + 1/2 B, 1/2 A + 1/2 B).

  Expected payoffs: (5,1), (1,5), (1/4 5 + 1/4 4 + 1/4 0 + 1/4 1 = 10/4 = 5/2,5/2).
- Suppose the players seek a mediator to help them. The mediator proposes the following:

  I'm going to toss a die. If it turns up either 1 or 2, will tell the Row player to play A, and otherwise I will tell them to play B. If it turns up either 5 or 6, will tell the Column player to play B, and otherwise I will tell them to play A.

  Do the players want to follow the advice?

Correlated Equilibrium: An Example

Coordination Game

		Col Player	
		A	B
Row Player	A	5,1	0,0
	B	4,4	1,5

The mediator's proposal:

> I'm going to toss a die. If it turns up either 1 or 2, will tell Row to play A, and otherwise I will tell them to play B. If it turns up either 5 or 6, will tell Column to play B, and otherwise I will tell them to play A.

- If Row is told to play A, they know die turned up $\{1, 2\}$. $\implies$ they also know Column will be told to play A half the times and B the remainder.
- Expected payoff: $1/2 \cdot 5 + 1/2 \cdot 0$.

  If they didn't follow the recommendation, then they'd get $1/2 \cdot 4 + 1/2 \cdot 1$; cannot do any better.

Correlated Equilibrium: An Example

Coordination Game

		Col Player	
		A	B
Row Player	A	5,1	0,0
	B	4,4	1,5

The mediator's proposal:

I'm going to toss a die. If it turns up either 1 or 2, will tell Row to play A, and otherwise I will tell them to play B. If it turns up either 5 or 6, will tell Column to play B, and otherwise I will tell them to play A.

- If Row is told to play B, they know die turned up $\{3, 4, 5, 6\}$. $\implies$ they also know Column will be told to play A half the times and B the remainder.
- Expected payoff: $1/2 \cdot 5 + 1/2 \cdot 0$.

  If they didn't follow the recommendation, then they'd get $1/2 \cdot 4 + 1/2 \cdot 1$; cannot do any better.
- Symmetric game: symmetric arguments apply for Column.
- Note: Row gets $1/3 \cdot [u_R(A, A) + u_R(B, A) + u_R(B, B)] = 1/310$; outside convex hull of NE payoffs.

Moral of the story: correlated eqm allows you to do more!

Overview

1. Motivation
2. Nash Equilibrium
3. Examples
4. Normal-Form Refinements and Generalisations of Nash Equilibrium
5. More

More

The originals: NE (Nash, 1950 PNAS, 1951), CE (Aumann, 1974 JMathEcon, 1987 Ecta), THPE (Selten, 1975 IJGT; Myerson, 1978 IJGT).

Beliefs and Epistemics: for NE Aumann & Brandenburger (1995 Ecta), for CE Brandenburger & Dekel (1987 Ecta).

Data-based and Belief Formation: CBDT (Gilboa, 1995 QJE).

MSNE in sports: (Walker & Wooders, 2001 AER; Palacios-Huerta, 2003 RES).

Experiments: BR to beliefs and incentives (Rey-Biel, 2009 GEB; Costa-Gomes & Weizsacker 2008 RES; Esteban-Casanelles & Gonçalves 2022 WP; Friedman & Ward 2024 WP)

Psychological Games: The role of emotions and intentions (Battigalli & Dufwenberg, 2019 JEL)

Payoffs and Social Preferences: Preferences over others' preferences (Ray & Vohra, 2019 AER); Inequality aversion (Fehr & Schmidt, 1999 QJE; Bolton & Ockenfels 2000 AER)

Level-k

WT incorporate reasoning mistakes.

Level-*k*

Cognitive Hierarchies

Endogenous Depth of Reasoning

Issues

(i) as if people have very unrealistic beliefs.

(ii) not well defined for arbitrary games.

(iii) "level" unstable even across dominance-solvable games.

(iv) individual's reasoning seems to depend on payoffs: take "more steps" of IESDS the higher the stakes.

(v) individual's reasoning seems to react to relative incentives smoothly.

Possible ways forward:

- pure stochastic choice as **Quantal Response Equilibrium** (McKelvey & Palfrey, 1995 GEB);
- model steps of reasoning via sampling **(Sequential) Sampling Equilibrium** (Osborne & Rubinstein 1998 AER, 2003 GEB; Gonçalves 2023 WP).