

11. Nash Equilibrium

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Overview

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Please predict choice frequencies.

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Now: Sharpening prediction, bridging the disconnect.

Goal: Nash equilibrium, GT's gold standard.

Used *everywhere*. (Not just economics or even social sciences.)

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1. Motivation
2. Nash Equilibrium
3. Examples
4. Normal-Form Refinements and Generalisations of Nash Equilibrium
5. More

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1. Motivation

2. Nash Equilibrium

- Definition and Interpretations
- Existence of a Nash Equilibrium
- Relation to Dominance
- Characterising Equilibria
- Interpreting MSNE
- Robustness

3. Examples

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5. More

Nash Equilibrium

Definition

$s \in S$ is a **pure strategy Nash equilibrium** iff $\forall i, u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i}) \forall s'_i \in S_i$ (s_i is BR to s_{-i}).

Note the difference: equilibrium, equilibrium payoff, equilibrium outcome.

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Non-degenerate mixed strategies $\neq$ totally mixed strategies.

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$\sigma \in \Sigma$ is a **Nash equilibrium** if $\forall i, u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma'_i, \sigma_{-i}) \forall \sigma'_i \in \Sigma_i$ (σ_i is BR to σ_{-i}).

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Player i 's best-response correspondence is given by $b_i : \Sigma_{-i} \rightrightarrows \Sigma_i$ s.t. $b_i(\sigma_{-i}) := \arg \max_{\sigma_i \in \Sigma_i} u_i(\sigma_i, \sigma_{-i})$.

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Remark

σ is a Nash equilibrium iff $\sigma \in b(\sigma)$.

Nash Equilibrium: Interpretation

(1) **Resulting from introspection.**

Everyone is BR to everyone else. If not, they'd prefer to do something else.

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Generally: What do to when there are multiple equilibria?

How does one decide which one is to be played?

Nash Equilibrium: Interpretation

- (1) **Resulting from introspection.**
- (2) **An outcome of learning, a steady state of a long-run adjustment process.**

Can help in selection of an equilibrium.

Fudenberg & Levine (1998); see also Fudenberg & Levine (2016) and Fudenberg (2022) for surveys.

Learning and dynamic adjustment: in next year's theory topics course!

Growing literature on estimating equilibria in games; what about dynamic adjustment toward equilibrium? (workshop on GT & metrics sponsored by cemmap)

Existence of a Nash Equilibrium

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Write down assumptions formally, but there is no equilibrium; your model is unable to make predictions!

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Existence results ensure that, under a given set of assumptions, your model *works* (whether it makes good predictions or not it is another matter)

Existence of a Nash Equilibrium

Kakutani's Fixed-Point Theorem

Let $X \subset \mathbb{R}^n$ be nonempty, compact, and convex. If $F : X \rightrightarrows X$ is nonempty-valued, compact-valued, convex-valued, and uhc, then $\exists x \in X : x \in F(x)$, i.e., there is a fixed point of F .

Existence of a Nash Equilibrium

Theorem

Let $\Gamma = \langle I, S, u \rangle$ be a normal-form game s.t. $|I| < \infty$, and, $\forall i \in I$, S_i is a nonempty, compact, and convex subset of $\mathbb{R}^n$. If $u_i : S \rightarrow \mathbb{R}$ is continuous in S and quasiconcave in s_i , then, there is a PSNE.

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Proof

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(i) b is nonempty-, compact-valued, and UHC.

u_i continuous, S_i is compact $\implies b_i$ nonempty-valued, compact-valued and UHC

$\forall i \in I$ (by Berge's maximum theorem).

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(ii) b is convex-valued.

u_i is quasiconcave in $s_i \implies b_i(s_{-i})$ is convex $\forall s_{-i} \in S_{-i}, \forall i \in I \implies b$ convex-valued.

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By Kakutani's fixed-point theorem, $\exists s \in b(s)$. □

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Let $\Gamma = \langle I, S, u \rangle$ be a normal-form game s.t. $|I|, |S| < \infty$. Then, there is a NE, possibly in mixed-strategies.

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Proof

- Game in mixed-strategies as a different game, $\tilde{\Gamma} = \langle I, \Sigma, \tilde{u} \rangle$, with $\tilde{u}_i(\sigma) = \mathbb{E}_{\sigma}[u_i]$.

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- Σ_i as a nonempty, compact, and convex subset of $[0, 1]^{|S_i|}$
- $\tilde{u}_i$ continuous in σ and linear (hence quasiconcave) in σ_i .
- Conditions of theorem met, hence $\exists$ PSNE σ of $\tilde{\Gamma}$, which is NE (possibly mixed) of Γ . $\square$

Existence of a Nash Equilibrium

Nash provided a different proof, based on Brouwer's fixed point theorem:

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Instructive proof by Geneakoplos.

Theorem

Let $\Gamma = \langle I, S, u \rangle$ be a normal-form game s.t. $|I| < \infty$, and, $\forall i \in I$, S_i is a nonempty, compact, and convex subset of $\mathbb{R}^n$. If $u_i : S \rightarrow \mathbb{R}$ is continuous in S and concave in s_i , then there is a PSNE.

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Proof

- Let $\phi_i : S \rightarrow S_i$ be s.t. $\phi_i(s) := \arg \max_{s'_i \in S_i} u_i(s'_i, s_{-i}) - \|s_i - s'_i\|^2$.

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- $u_i(s'_i, s_{-i})$ is concave in s'_i and $-\|s_i - s'_i\|^2$ is strictly concave
 $\implies u_i(s'_i, s_{-i}) - \|s_i - s'_i\|^2$ strictly concave and continuous in s'_i .

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- Berge's maximum theorem $\implies \phi_i$ UHC + singleton-valued $\implies \phi_i$ continuous.
- Let $\phi(s) := (\phi_i(s))_{i \in I}$. $\phi : S \rightrightarrows S$ continuous, $S \subset \mathbb{R}^n$ convex, compact $\implies \phi$ has fixed point $s = \phi(s)$ (by Brouwer's fixed-point theorem).

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 - u_i concave $\implies u_i(\alpha s'_i + (1 - \alpha)s_i, s_{-i}) - u_i(s) \geq \alpha(u_i(s'_i, s_{-i}) - u_i(s)) \forall \alpha \in (0, 1)$.
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$$\begin{aligned} 0 &\geq u_i((\alpha s'_i + (1 - \alpha)s_i), s_{-i}) - \|s_i - (\alpha s'_i + (1 - \alpha)s_i)\|^2 \\ &\quad - \left(\max_{s''_i \in S_i} u_i(s''_i, s_{-i}) - \|s_i - s''_i\|^2 \right) \end{aligned}$$

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$$\begin{aligned} 0 &\geq u_i((\alpha s'_i + (1 - \alpha)s_i), s_{-i}) - \|s_i - (\alpha s'_i + (1 - \alpha)s_i)\|^2 \\ &\quad - \left(\max_{s''_i \in S_i} u_i(s''_i, s_{-i}) - \|s_i - s''_i\|^2 \right) \\ &= u_i((\alpha s'_i + (1 - \alpha)s_i), s_{-i}) - \|s_i - (\alpha s'_i + (1 - \alpha)s_i)\|^2 - u_i(s) \end{aligned}$$

Existence of a Nash Equilibrium

Proof

- $\phi_i(s) := \arg \max_{s'_i \in S_i} u_i(s'_i, s_{-i}) - \|s_i - s'_i\|^2$. $\phi(s) := (\phi_i(s))_{i \in I}$. $\exists s \in S : s = \phi(s)$.
- WTS $s_i \in \arg \max_{s_i \in S_i} u_i(s_i, s_{-i})$.
 - Suppose not, i.e., $\exists s'_i \in S_i : u_i(s'_i, s_{-i}) > u_i(s)$ for some i .
 - u_i concave $\implies u_i(\alpha s'_i + (1 - \alpha)s_i, s_{-i}) - u_i(s) \geq \alpha(u_i(s'_i, s_{-i}) - u_i(s)) \forall \alpha \in (0, 1)$.
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- Choosing α s.t. $\alpha < (u_i(s'_i, s_{-i}) - u_i(s)) / \|s_i - s'_i\|^2$ delivers

$$0 \geq \alpha(u_i(s'_i, s_{-i}) - u_i(s)) - \alpha^2 \|s_i - s'_i\|^2 > \alpha^2 - \alpha^2 = 0. \text{ a contradiction.}$$

□

Symmetric Nash Equilibria

Definition

Let $\Gamma = \langle I, S, u \rangle$ be a normal-form game. Γ is **symmetric** iff $\forall i, j \in I, S_j = S_i$, and $u_i(s_i, s_{-i}) = u_j(s_j, s_{-j})$ for $s_i = s_j$ and $s_{-i} = s_{-j}$.

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Let $\Gamma = \langle I, S, u \rangle$ be a symmetric normal-form game s.t. $|I| < \infty$, and, $\forall i \in I, S_i$ is a nonempty, compact, and convex subset of $\mathbb{R}^n$. If $u_i : S \rightarrow \mathbb{R}$ is continuous in S and quasiconcave in s_i , then there is a PSNE s.t. $s_i = s_j \in S_i \forall i, j \in I$.

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Proof

Define $b : S_i \rightrightarrows S_i$ s.t. $\tilde{b}(s_i) = b_i(s_{-i})$ for $s_{-i} = (s_j)_{j \in -i}$; b_i is player i 's best-response correspondence.

$\tilde{b}$ nonempty-, compact-, and convex-valued, and UHC
 $\implies$ Kakutani's fixed-point theorem applies. □

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Theorem

Let X be a nonempty, compact, convex subset of a locally convex Hausdorff (e.g., vector) space and that $f : X \rightrightarrows X$ is nonempty- and convex-value correspondence with a closed graph. Then $\exists x \in X : x \in f(x)$.

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Does 2PA have continuous payoffs? What to do then? See Reny (1999 Ecta) "On the Existence of Pure and Mixed Strategy Nash Equilibria in Discontinuous Games"

Relation to Dominance

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- (ii) Any Nash equilibrium strategy must be rationalisable (and thus survive IESDS).
- (iii) Any pure strategy in the support of a Nash equilibrium is also rationalisable.
- (iv) However... weakly dominated strategies *can* be played with positive probability at a Nash equilibrium.

Relation to Dominance

		Col Player	
		A	B
Row Player	A	1,1	0,0
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NE: (A,A) and (B,B)

Characterising Equilibria

		Col Player	
		A	B
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PSNE: (A,B) and (B,A)

Characterising Equilibria

Remark

σ is a Nash equilibrium if and only if $\forall i \in I$, (i) $u_i(s_i, \sigma_{-i}) \geq u_i(s'_i, \sigma_{-i}) \forall s_i \in \text{supp}(\sigma_i), s'_i \in S_i$, and (ii) $u_i(s_i, \sigma_{-i}) = u_i(s'_i, \sigma_{-i}) \forall s_i, s'_i \in \text{supp}(\sigma_i)$.

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Best response condition: any pure strategy in the support must be best response to σ_{-i} ,
 $u_i(s_i, \sigma_{-i}) \geq u_i(s'_i, \sigma_{-i})$ for any $s_i \in \text{supp}(\sigma_i)$ and $s'_i \in S_i$.

MSNE indifference condition: must get same payoff $u_i(s_i, \sigma_{-i}) = u_i(s'_i, \sigma_{-i})$ for any pure strategy in the support, $s_i, s'_i \in \text{supp}(\sigma_i)$.

Characterising Equilibria

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PSNE: (A,B) and (B,A)

Given σ_C ,

$$u_R(A, \sigma_C) \geq u_R(B, \sigma_C) \implies \sigma_C(A)1 + (1 - \sigma_C(A))2 \geq \sigma_C(A)3 + (1 - \sigma_C(A))0$$

$$\sigma_C(A) \leq \frac{1}{2}.$$

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NE: (A,B), (B,A), and $(\sigma_R, \sigma_C) : \sigma_R(A) = \sigma_C(A) = 1/2$.

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As stochastic choice: players look like they are randomising, but could be random utility players ('purification' e.g. Harsanyi 1973 IJGT – more later), unobserved information acquisition (Gonçalves 2024 WP), etc.

MSNE *can be* rationalised as the limit outcome of one such situation.

Robustness

Will limit of equilibria be an equilibrium of the limit game? (is set of NE UHC?)

Proposition

Let $S := \times_{i \in I} S_i$ be such that S_i is nonempty, compact, and convex subset of $\mathbb{R}^{n_i}$, $T \subseteq \mathbb{R}^m$, $u_i : S \times T \rightarrow \mathbb{R}$. Let $S^{NE}(t) := \left\{ s \in S \mid s_i \in \arg \max_{s'_i \in S_i} u_i(s'_i, s_{-i}, t) \right\}$ be set of NE of the game $\Gamma_t := \langle I, S, u^t \rangle$, where $u^t := (u_i(\cdot, t))_{i \in I}$.

If (i) u_i is continuous in (s, t) , and (ii) $S^{NE}(t')$ is nonempty for any t' in a neighborhood of t , then S^{NE} is UHC at t .

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Proof

- Take $(s^n, t^n) \rightarrow (s, t)$, where $s^n \in S^{NE}(t^n)$ for all n .

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- Then

$$u_i(s_i, s_{-i}, t) = \lim_{n \rightarrow \infty} u_i(s_i^n, s_{-i}^n, t^n) = \lim_{n \rightarrow \infty} \max_{s'_i \in S_i} u_i(s'_i, s_{-i}^n, t^n) = \max_{s'_i \in S_i} u_i(s'_i, s_{-i}, t).$$

- Hence, $s \in S^{NE}(t)$ and S^{NE} is uhc (and compact-valued) at t . □

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Issue: not all NE are robust; small mistakes may 'kill fragile equilibria'. More later.

Overview

1. Motivation

2. Nash Equilibrium

3. Examples

- Common Value All-Pay Auction
- Model of Sales

4. Normal-Form Refinements and Generalisations of Nash Equilibrium

5. More

Common Value All-Pay Auction

I bidders, all value object at $v > 0$. Bids $s_i \geq 0$.

Payoffs: always pay bid; win if bid highest; ties broken uniformly at random.

$$u_i(s_i, s_{-i}) = \mathbf{1}\{s_i = \max_j s_j\} \cdot \frac{1}{|\{j \in I \mid s_j = \max_{\ell} s_{\ell}\}|} v - s_i$$

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Claim 1: No one bids above v : strictly dominated.

Claim 2: No PSNE in this game.

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- (c) If $\max_j s_j = v$, $i, \ell \in \arg \max_j s_j$ with $i \neq \ell$, then i wants to deviate to $s'_i = 0$.

Common Value All-Pay Auction

Is there NE?

Common Value All-Pay Auction

Is there NE? Let's try to construct a symmetric MSNE $\sigma_i = \sigma_j$ (CDF):

- NB: $\sigma_i \in \Delta([0, v])$; (why?)

Common Value All-Pay Auction

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- Conclusion: Competition left bidders with 0 expected surplus from the auction.

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Two firms sell a good to a unit mass of consumers with reservation price of £1 (WTP).

Firms set prices simultaneously.

Each firm has loyal consumers (insofar as price doesn't exceed reservation price):

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- (v) Pricing below 1/3 yields $\frac{3}{4}p_i < \frac{1}{4}$.
(i.e., in expectation worse than pricing at $p_i \in [1/3, 1]$.)

Overview

1. Motivation
2. Nash Equilibrium
3. Examples
4. Normal-Form Refinements and Generalisations of Nash Equilibrium
 - Trembling-Hand Perfection
 - Correlated Equilibrium
5. More

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Idea: non-zero probabilities on each pure strategy capture the notion of unavoidable mistakes.

Define $\Delta_{\epsilon}(S_i) := \{\sigma_i \in \Sigma_i \mid \sigma_i(s_i) \geq \epsilon(s_i), \forall s_i \in S_i\}$ for $\epsilon : \cup_{i \in I} S_i \rightarrow (0, 1)$.

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Remark

For any ϵ , $\Delta_\epsilon(S_i)$ is compact. Hence, insofar as $\Delta_\epsilon(S_i)$ is nonempty for all i , there is an ϵ -constrained NE.

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Take any sequence of ϵ^n -constrained equilibrium for $n > N$; as it lives in a compact set, it admits a convergent subsequence.

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Take any sequence of ϵ^n -constrained equilibrium for $n > N$; as it lives in a compact set, it admits a convergent subsequence.

As $u_i : \Sigma \rightarrow \mathbb{R}$ is continuous $\forall i$, subsequence converges to a NE of original game.

Trembling-Hand Perfection

Proposition

A NE σ of game Γ is THPE iff $\exists$ sequence of fully mixed strategy profiles $\sigma^n \rightarrow \sigma : \forall i$ and n , σ_i is a best response to σ_{-i}^n .

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- But then $\sigma_i^k(s_i) \rightarrow 0$, which contradicts the fact that $\sigma_i^k(s_i) \rightarrow \sigma_i(s_i) > 0$.

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and therefore σ_i^n is ε^n -constrained BR to σ_{-i}^n . (Feasible $\forall n > 2|S_i|$)

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Other notions: strong or strict equilibrium

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And Now for Something Completely Different...

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Definition

A probability distribution $p \in \Delta(S)$ is a **correlated equilibrium** of a normal-form game $\Gamma = \langle I, S, u \rangle$ if $\forall i$ and $\forall s_i : p(s_i) > 0$

$$\sum_{s_{-i} \in S_{-i}} p(s_{-i} | s_i) u_i(s_i, s_{-i}) \geq \sum_{s_{-i} \in S_{-i}} p(s_{-i} | s_i) u_i(s'_i, s_{-i}), \quad \forall s'_i \in S_i.$$

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Corollary

CE exists in finite games

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Usefulness: Easily computable. Mediator trying to get an outcome to emerge.
(Example: advertising a mega party.)

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Convex hull of set of NE is subset of set of CE (also convex).

$\implies$ Can attain any convex combination of NE payoffs.

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A probability distribution $p \in \Delta(S)$ is a **correlated equilibrium** of a normal-form game $\Gamma = \langle I, S, u \rangle$ if $\forall i$ and $\forall s_i : p(s_i) > 0$

$$\sum_{s_{-i} \in S_{-i}} p(s_{-i} | s_i) u_i(s_i, s_{-i}) \geq \sum_{s_{-i} \in S_{-i}} p(s_{-i} | s_i) u_i(s'_i, s_{-i}), \quad \forall s'_i \in S_i.$$

Usefulness: Easily computable. Mediator trying to get an outcome to emerge.

(Example: advertising a mega party.)

Convex hull of set of NE is subset of set of CE (also convex).

$\implies$ Can attain any convex combination of NE payoffs. That's it?

Correlated Equilibrium: An Example

Coordination Game

		Col Player	
		A	B
Row Player	A	5,1	0,0
	B	4,4	1,5

- NE?

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- NE? (A,A), (B,B), and $(\frac{1}{2} A + \frac{1}{2} B, \frac{1}{2} A + \frac{1}{2} B)$.

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Expected payoffs: (5,1), (1,5), $(\frac{1}{4} 5 + \frac{1}{4} 4 + \frac{1}{4} 0 + \frac{1}{4} 1 = 10/4 = 5/2, 5/2)$.

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- Suppose the players seek a mediator to help them. The mediator proposes the following:

I'm going to toss a die. If it turns up either 1 or 2, will tell the Row player to play A, and otherwise I will tell them to play B. If it turns up either 5 or 6, will tell the Column player to play B, and otherwise I will tell them to play A.

Do the players want to follow the advice?

Correlated Equilibrium: An Example

Coordination Game

		Col Player	
		A	B
Row Player	A	5,1	0,0
	B	4,4	1,5

The mediator's proposal:

I'm going to toss a die. If it turns up either 1 or 2, will tell Row to play A, and otherwise I will tell them to play B. If it turns up either 5 or 6, will tell Column to play B, and otherwise I will tell them to play A.

- If Row is told to play A, they know die turned up $\{1, 2\}$. $\implies$ they also know Column will be told to play A half the times and B the remainder.
- Expected payoff: $1/2 \cdot 5 + 1/2 \cdot 0$.

If they didn't follow the recommendation, then they'd get $1/2 \cdot 4 + 1/2 \cdot 1$; cannot do any better.

Correlated Equilibrium: An Example

Coordination Game

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		A	B
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- If Row is told to play B, they know die turned up $\{3, 4, 5, 6\}$. $\implies$ they also know Column will be told to play A half the times and B the remainder.
- Expected payoff: $1/2 \cdot 5 + 1/2 \cdot 0$.

If they didn't follow the recommendation, then they'd get $1/2 \cdot 4 + 1/2 \cdot 1$; cannot do any better.

- Symmetric game: symmetric arguments apply for Column.
- Note: Row gets $1/3 \cdot [u_R(A, A) + u_R(B, A) + u_R(B, B)] = 1/3 \cdot 10$; outside convex hull of NE payoffs.

Correlated Equilibrium: An Example

Coordination Game

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The mediator's proposal:

I'm going to toss a die. If it turns up either 1 or 2, will tell Row to play A, and otherwise I will tell them to play B. If it turns up either 5 or 6, will tell Column to play B, and otherwise I will tell them to play A.

- If Row is told to play B, they know die turned up $\{3, 4, 5, 6\}$. $\implies$ they also know Column will be told to play A half the times and B the remainder.
- Expected payoff: $1/2 \cdot 5 + 1/2 \cdot 0$.

If they didn't follow the recommendation, then they'd get $1/2 \cdot 4 + 1/2 \cdot 1$; cannot do any better.

- Symmetric game: symmetric arguments apply for Column.
- Note: Row gets $1/3 \cdot [u_R(A, A) + u_R(B, A) + u_R(B, B)] = 1/310$; outside convex hull of NE payoffs.

Moral of the story: correlated eqm allows you to do more!

Overview

1. Motivation
2. Nash Equilibrium
3. Examples
4. Normal-Form Refinements and Generalisations of Nash Equilibrium
5. More

More

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Payoffs and Social Preferences: Preferences over others' preferences (Ray & Vohra, 2019 AER); Inequality aversion (Fehr & Schmidt, 1999 QJE; Bolton & Ockenfels 2000 AER)

Level- k

WT incorporate reasoning mistakes.

Level- k

Cognitive Hierarchies

Endogenous Depth of Reasoning

Issues

- (i) as if people have very unrealistic beliefs.
- (ii) not well defined for arbitrary games.
- (iii) “level” unstable even across dominance-solvable games.
- (iv) individual’s reasoning seems to depend on payoffs: take “more steps” of IESDS the higher the stakes.
- (v) individual’s reasoning seems to react to relative incentives smoothly.

Possible ways forward:

pure stochastic choice as **Quantal Response Equilibrium** (McKelvey & Palfrey, 1995 GEB);

model steps of reasoning via sampling (**Sequential) Sampling Equilibrium** (Osborne & Rubinstein 1998 AER, 2003 GEB; Gonçalves 2023 WP).