

12. Incomplete Information

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Overview

Complete information assumption implies players know others' payoffs.

Examples:

- goal keeper may not really know how effortful it is for the penalty kicker to shoot right instead of left;
- firms may not know other firms' cost structure;
- voters may not know how other voters' preferences;
- consumers may be unsure of how much they value a good;
- investors may not know what is the value of an asset;
- firm may not know how productive a given job candidate is;
- a researcher may not know how difficult a problem they're working on is.

Today's agenda: formalising games of incomplete information and examining applications.

Overview

1. Motivation
2. Bayesian Games
3. Bayesian Nash Equilibrium
4. Auctions
5. Purification Theorem
6. Higher-Order Beliefs
7. More

Overview

1. Motivation

2. Bayesian Games

- Representing Incomplete Information

3. Bayesian Nash Equilibrium

4. Auctions

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Representing Incomplete Information

Definition

A game is of **incomplete information** when at least one player does not know the payoff that some player receives from some strategy profile.

How to model uncertainty?

Harsanyi's modelling insight:

Transform incomplete info game into complete info with Nature moving at start of game.

Realisation of nature's actions determines players' payoffs.

Assumption: CK of prob. distrib. used by Nature.

Players have a belief about others' preferences and there is common knowledge of such beliefs.

Representing Incomplete Information

Definition

A **Bayesian game** is a tuple $\langle I, A, u, \Theta, \mathbf{p} \rangle$, where

- (i) Players: I ;
- (ii) Player i 's action space A_i ; Space of action profiles $a \in A := \times_{i \in I} A_i$;
- (iii) Player i 's type space Θ_i ; Space of type profiles $\Theta := \times_{i \in I} \Theta_i$;
- (iv) Player i 's utility/payoff function: $u_i : A \times \Theta \rightarrow \mathbb{R}$; $u := (u_i)_{i \in I}$; and
- (v) Probability distribution over players' type profiles: $\mathbf{p} \in \Delta(\Theta)$.

All elements are common knowledge, but each player i only knows their own type θ_i , and not the other players' types.

Players privately learn their own type. (WLOG)

Representing Incomplete Information

Definition

- Players have **private values** iff $u_i(a, \theta_i, \theta_{-i}) = u_i(a, \theta_i, \theta'_{-i}) \forall \theta_{-i}, \theta'_{-i} \in \Theta_{-i}$. Otherwise, they have **interdependent values**.
- Players have **independent types** iff types are independent across players. Otherwise, they have **correlated types**.

Could also consider alternative notions of incomplete information: e.g., uncertainty over what is the strategy set of the opponent.

Representing Incomplete Information

Definition

A pure **strategy** of player i in a Bayesian game is a mapping $s_i : \Theta_i \rightarrow A_i$.

Strategy specifies action for each possible type.

Player i 's expected payoff: $\tilde{u}_i(s) = \mathbb{E}_{\theta \sim p}[u_i(s_1(\theta_1), s_2(\theta_2), \dots, s_i(\theta_i), \theta_i, \theta_{-i})]$.

Extend $\tilde{u}_i$ to mixed strategies, $\sigma_i \in \Sigma_i := \Delta(S_i)$.

Representing Incomplete Information

Two classmates, A and B, considering whether to work together.

They work together iff both agree to do so.

If they work alone, payoffs normalised to 0.

If they work together B always gets 10 (improves their grade by 10). However, how much A benefits from working with B depends on B's type.

If B is collaborative (wp α), A also gets a payoff of 10. But if B is a shirker (wp $1 - \alpha$), then A gets a payoff of -6.

If $\alpha = 1$ or $\alpha = 0$, what are the NE?

What are the strategies of this Bayesian game?

Table: $\theta_B = C$

		B	
		W	N
A	W	10, 10	0, 0
	N	0, 0	0, 0

Table: $\theta_B = S$

		B	
		W	N
A	W	-6, 10	0, 0
	N	0, 0	0, 0

Universal Type Space

Are Bayesian games sufficiently rich to capture all kinds of incomplete information?

Higher-order uncertainty and belief hierarchy

- Uncertainty about others' preferences
- Uncertainty about others' uncertainty about one's preferences
- Uncertainty about others' uncertainty about one's uncertainty about others' preferences
- Uncertainty about

Type should capture the *entire* belief hierarchy.

Can we capture rich uncertainty just with set of types and distribution over types?

Yes, with a **universal type space** (Mertens & Zamir (1985); Bradenburger & Dekel (1993))

Reassuring that Bayesian games are good tool.

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Bayesian Nash Equilibrium

Definition

A **Bayesian Nash Equilibrium** of a Bayesian game $\langle I, A, u, \Theta, \rho \rangle$ is a strategy profile $s = (s_i)_{i \in I}$ such that $\forall i, \forall s'_i \in S_i, \tilde{u}_i(s_i, s_{-i}) \geq \tilde{u}_i(s'_i, s_{-i})$.

NB: consider Bayesian game $\Gamma = \langle I, A, u, \Theta, \rho \rangle$ as standard normal-form game $\tilde{\Gamma} = \langle I, S, \tilde{u} \rangle$.

Set of BNE of Γ is the same as set of NE of $\tilde{\Gamma}$.

Can tweak NE existence theorems to work for BNE.

Corollary

For any Bayesian game Γ s.t. $|I|, |A|, |\Theta| < \infty$, $\exists$ Bayesian Nash equilibrium (possibly in mixed strategies).

Ex-ante vs Interim perspective

Ex-ante Perspective:

1. players choose strategies, (distrib. over) mappings from types to actions, to maximise ex-ante expected payoff;
2. types are drawn according to $\mathbf{p}$;
3. players learn their own types and play according to their actions;
4. outcomes and payoffs realise.

Interim perspective:

1. types are drawn according to $\mathbf{p}$;
2. players learn their own types, form beliefs about others' types $q_i(\cdot \mid \boldsymbol{\theta}_i)$, and play according to their actions;
3. players choose (distrib. over) actions, to maximise (ex-interim) expected payoff, knowing their type, but not opponents' types;
4. outcomes and payoffs realise.

Arguably more sensible description of a game of incomplete information.

Definition

An ex-interim **Bayesian game** is a tuple $\langle I, A, u, \Theta, q \rangle$, where

- (i) Players: I ;
- (ii) Player i 's action space A_i ; Space of action profiles $a \in A := \times_{i \in I} A_i$;
- (iii) Player i 's type space Θ_i ; Space of type profiles $\Theta := \times_{i \in I} \Theta_i$;
- (iv) Player i 's utility/payoff function: $u_i : A \times \Theta \rightarrow \mathbb{R}$; $u := (u_i)_{i \in I}$; and
- (v) Ex-interim Belief/Prob. distrib. over opponents' type profiles: $q_i : \Theta_{-i} \rightarrow \Delta(\Theta_{-i})$.

Ex-Ante vs Interim perspective

Proposition

A strategy profile σ is a BNE if and only if $\forall i \in I$ and $\forall \theta_i \in \Theta_i : p(\theta_i) > 0$,

$$\mathbb{E}_{\theta_{-i}}[u_i(\sigma_i(\theta_i), \sigma_{-i}(\theta_{-i}), \theta_i, \theta_{-i}) \mid \theta_i] \geq \mathbb{E}_{\theta_{-i}}[u_i(\sigma'_i(\theta_i), \sigma_{-i}(\theta_{-i}), \theta_i, \theta_{-i}) \mid \theta_i], \quad \forall \sigma'_i \in \Delta(A_i)^{\Theta_i}.$$

NB: Possible to find $p_i \in \Delta(\Theta) : p_i(\theta_{-i} \mid \theta_i) = q_i(\theta_{-i} \mid \theta_i) \quad \forall \theta_i, \theta_{-i}$
(going beyond finite case introduces technical complications).

However: players may not start with **common prior**: $p_i = p_j$ for all i, j .

Common prior necessary for equivalence between ex-ante BNE and interim BNE.

Ex-Post Bayesian Nash Equilibrium

Definition

A strategy profile σ is an **ex-post Bayesian Nash equilibrium** iff $\forall i, u_i(\sigma_i(\theta_i), \sigma_{-i}(\theta_{-i}), \theta) \geq u_i(a_i, \sigma_{-i}(\theta_{-i}), \theta), \forall a_i, \theta$.

Interpretation: Even if players learn others' types, they would not like to change their actions, given that others are following their strategies.

Note: ex-post BNE yields NE for each game indexed by θ .

True or false? There is always an ex-post BNE.

Table: $\theta_B = C$

		B	
		W	N
A	W	10, 10	10, 0
	N	0, 1	0, 0

Table: $\theta_B = S$

		B	
		W	N
A	W	-1, 10	-1, 0
	N	0, 1	0, 0

Strategy-Proofness

Closely related to “**Very weak dominance**”: $s_i : u_i(s_i(\theta_i), a_{-i}, \theta_i, \theta_{-i}) \geq \tilde{u}_i(a_i, s_{-i}, \theta_i, \theta_{-i}),$
 $\forall a_i, a_{-i}, \forall \theta.$

Allows for indifferences.

Also said **Strategy-proofness**, esp. when $A_i = \Theta_i$.

You’ll hear this term a lot.

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- 2nd-Price Auction
- Envelope Theorem
- 1st-Price Auction
- Revenue Equivalence

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2nd-Price Auction

2nd-Price Auction: winner pays second highest bid.

$$u_i(a_i, a_{-i}, v_i) = \mathbf{1}\{i \in \arg \max_j a_j\}(v_i - \max_{j \neq i} a_j) / |\arg \max_j a_j|$$

When F_i is degenerate for every i , $a_i = v_i$ is weakly dominant for all players (hence a NE?).

Independent private values. (What does this mean?)

$v_i \sim F_i$, v_i independent from other types.

$s_i : s_i(v_i) = v_i$ still weakly dominant for all players? Is it a BNE? What does it depend on?

Informationally robust (although perhaps counterintuitive to people.)

Alternative: Ascending auction. People understand it better and play weakly dominant strategy more often.

With good reasons: [Obviously Strategy-Proof](#) (Li, 2017 AER)

Roughly, worst case scenario better than best-case scenario from deviation.

Envelope Theorem

What is the envelope theorem?

Relate effect of parameter on value function to its effect on the objective function.

Useful tool to characterise how maximisers change with parameters too!

Choice set X .

Parameter $t \in [0, 1]$ (think directional derivative in normed vector space)

Objective function: $f : X \times [0, 1] \rightarrow \mathbb{R}$.

Value function: $V(t) := \sup_{x \in X} f(x, t)$; Maximisers $X^*(t) := \{x \in X : f(x, t) = V(t)\}$.

Theorem 1 (Milgrom & Segal 2002 Ecta)

Take any $x^* \in X^*(t)$ and $t \in [0, 1]$, and suppose $f'_t(x^*, t)$ exists.

- (1) For $t > 0$, if V is left-differentiable at t , $V'(t^-) \leq f'_t(x^*, t)$.
- (2) For $t < 1$, if V is right-differentiable at t , $V'(t^+) \geq f'_t(x^*, t)$.
- (3) For $t \in (0, 1)$, if V is differentiable at t , then $V'(t) = f'_t(x^*, t)$.

Envelope Theorem

It would be sufficient to ensure V is differentiable a.e. to get

Theorem 2 (Milgrom & Segal 2002 Ecta)

Take any $x^* \in X^*(t)$ and $t \in [0, 1]$, and suppose $f'_t(x^*, t)$ exists.

- (1) If $f(x, \cdot)$ is absolutely continuous for all $x \in X$ and there is an integrable function $b : [0, 1] \rightarrow \mathbb{R}_+$ such that $|f'_t(x, t)| \leq b(t) \forall x \in X$ and almost all $t \in [0, 1]$, then V is absolutely continuous.
- (2) If, in addition, $f(x, \cdot)$ is differentiable for all $x \in X$ and X^* is nonempty-valued a.e. on $[0, 1]$, then for any selection $x^*(t) \in X^*(t)$,

$$V(t) = V(0) + \int_0^t f'_t(x^*(s), s) ds$$

Note: V need not be differentiable everywhere (may have kinks).

1st-Price Auction

Back to auctions: **1st-Price Auction**: winner pays highest bid.

I bidders with valuations $0 \leq v_i$ and $v_i \sim F$ iid, F atomless and absolutely continuous, bounded support $V_i = [\underline{v}, \bar{v}]$

Bids: $a_i \geq 0$. **Strategies:** $s_i : V_i \rightarrow \mathbb{R}_+$

Payoffs

$$u_i(a_i, a_{-i}, v_i) := \mathbf{1}_{a_i \in \max_{j \in I} \{a_j\}} \frac{1}{|\arg \max_{j \in I} \{a_j\}|} (v_i - a_i)$$

Get zero if do not bid highest.

Get item if bid highest and pay own bid; uniform tie-breaking.

NB: $s_i(v_i) = v_i$ is weakly dominated in 1PA!

1st-Price Auction

Solving for a symmetric PS-BNE $(s^*)_{j \in I}$

Assume: s^* is strictly increasing, differentiable.

s^* strictly increasing + F atomless $\implies$ zero prob. of two identical bids.

Expected payoff from bidding a_i given type v_i and opponents bidding according to s^* :

$$\begin{aligned}u(a_i, v_i) &= \mathbb{P}(a_i > \max_{j \neq i} s^*(v_j))(v_i - a_i) \\&= \mathbb{P}(a_i > s^*(v_j))^{|I|-1}(v_i - a_i) \\&= F((s^*)^{-1}(a_i))^{|I|-1}(v_i - a_i).\end{aligned}$$

1st-Price Auction

Solving for a symmetric PS-BNE $(s^*)_{j \in I}$

Assume: s^* is strictly increasing, differentiable.

s^* BR to s^* :

$$\begin{aligned} u(a_i, v_i) &= F((s^*)^{-1}(a_i))^{|I|-1}(v_i - a_i). \\ \implies U(v_i) &:= u(s^*(v_i), v_i) = F((s^*)^{-1}(s^*(v_i)))^{|I|-1}(v_i - s^*(v_i)) \\ U(v_i) &= F(v_i)^{|I|-1}(v_i - s^*(v_i)). \end{aligned} \tag{1}$$

1st-Price Auction

Solving for a symmetric PS-BNE $(s^*)_{j \in I}$

Assume: s^* is strictly increasing, differentiable. (check later)

s^* strictly increasing $\implies s^*(\underline{v})$ wins auction wp0 $\implies U(\underline{v}) = 0$.

(a) $u(a_i, v_i) = F((s^*)^{-1}(a_i))^{I-1}(v_i - a_i)$ differentiable in v_i .

(b) $U(v_i) = F(v_i)^{I-1}(v_i - s^*(v_i))$ differentiable.

Use envelope theorem:

$$U'(v_i) = u'_{v_i}(a_i, v_i)|_{a_i=s^*(v_i)} = F((s^*)^{-1}(a_i))^{I-1}|_{a_i=s^*(v_i)} = F(v_i)^{I-1}. \quad (2)$$

Fundamental theorem of calculus:

$$U(v_i) = U(\underline{v}) + \int_{\underline{v}}^{v_i} U'(v) dv = \int_{\underline{v}}^{v_i} U'(v) dv = . \quad (3)$$

1st-Price Auction

Solving for a symmetric PS-BNE $(s^*)_{j \in I}$

Putting it all together:

$$U(v_i) = F(v_i)^{|I|-1}(v_i - s^*(v_i)). \quad (1)$$

$$U'(v_i) = F(v_i)^{|I|-1}. \quad (2)$$

$$U(v_i) = \int_{\underline{v}}^{v_i} U'(v) dv. \quad (3)$$

$$\implies \int_{\underline{v}}^{v_i} F(v)^{|I|-1} dv = F(v_i)^{|I|-1}(v_i - s^*(v_i))$$

$$\iff s^*(v_i) = v_i - \int_{\underline{v}}^{v_i} \left(\frac{F(v)}{F(v_i)} \right)^{|I|-1} dv$$

1st-Price Auction

Solving for a symmetric PS-BNE $(s^*)_{j \in I}$

$$s^*(v_i) = v_i - \int_{\underline{v}}^{v_i} \left(\frac{F(v)}{F(v_i)} \right)^{|I|-1} dv$$

Properties of s^*

Strictly increasing

$$\begin{aligned} & s^*(v' + e) - s^*(v') \\ &= e - \int_{\underline{v}}^{v'+e} \left(\frac{F(v)}{F(v' + e)} \right)^{|I|-1} dv + \int_{\underline{v}}^{v'} \left(\frac{F(v)}{F(v')} \right)^{|I|-1} dv \\ &= \int_{v'}^{v'+e} 1 dv - \int_{\underline{v}}^{v'+e} \left(\frac{F(v)}{F(v' + e)} \right)^{|I|-1} dv + \int_{\underline{v}}^{v'} \left(\frac{F(v)}{F(v')} \right)^{|I|-1} \left(\frac{F(v' + e)}{F(v')} \right)^{|I|-1} dv \\ &= \int_{v'}^{v'+e} 1 dv - \int_{v'}^{v'+e} \left(\frac{F(v)}{F(v' + e)} \right)^{|I|-1} dv + \int_{\underline{v}}^{v'} \left(\frac{F(v)}{F(v')} \right)^{|I|-1} \frac{F(v' + e)^{|I|-1} - F(v')^{|I|-1}}{F(v' + e)^{|I|-1}} dv \\ &\geq \int_{v'}^{v'+e} \frac{F(v' + e)^{|I|-1} - F(v)^{|I|-1}}{F(v' + e)^{|I|-1}} dv + \int_{\underline{v}}^{v'} \left(\frac{F(v)}{F(v')} \right)^{|I|-1} \frac{F(v' + e)^{|I|-1} - F(v')^{|I|-1}}{F(v' + e)^{|I|-1}} dv > 0. \end{aligned}$$

1st-Price Auction

Solving for a symmetric PS-BNE $(s^*)_{j \in I}$

$$s^*(v_i) = v_i - \int_{\underline{v}}^{v_i} \left(\frac{F(v)}{F(v_i)} \right)^{|I|-1} dv$$

Properties of s^*

Strictly increasing.

Differentiable (immediate).

Bid less than value $s^*(v_i) < v_i$ for $v_i > \underline{v}$. $\implies U(v_i) \geq 0$.

1st-Price Auction

Solving for a symmetric PS-BNE $(s^*)_{j \in I}$

$$s^*(v_i) = v_i - \int_{\underline{v}}^{v_i} \left(\frac{F(v)}{F(v_i)} \right)^{I-1} dv$$

WT check s^* is optimal.

Take any $v_i \in V_i$ and $a_i \geq 0$.

Claim: Given others play s^* , $s^*(v_i)$ does weakly better than $a_i \forall a_i \notin [s^*(\underline{v}), s^*(\bar{v})]$.

If $a_i < \underline{v} = s^*(\underline{v})$, then $0 = u(a_i, v_i) = U(\underline{v}) \leq U(v_i)$.

If $a_i > s^*(\bar{v})$, then $u(a_i, v_i) = v_i - a_i < v_i - s^*(\bar{v})$.

1st-Price Auction

Solving for a symmetric PS-BNE $(s^*)_{j \in I}$

$$s^*(v_i) = v_i - \int_{\underline{v}}^{v_i} \left(\frac{F(v)}{F(v_i)} \right)^{|I|-1} dv$$

WT check s^* is optimal.

Take any $v_i \in V_i$ and $a_i \geq 0$.

Claim: Given others play s^* , $s^*(v_i)$ does weakly better than $a_i \forall a_i \in [s^*(\underline{v}), s^*(\bar{v})]$.

NB: s^* continuous and strictly increasing $\implies \exists! v'_i : a_i = s^*(v'_i) \forall a_i \in [s^*(\underline{v}), s^*(\bar{v})]$.

$$\begin{aligned} U(v_i) - u(a_i, v_i) &= U(v_i) - U(v'_i) + U(v'_i) - u(s^*(v'_i), v_i) \\ &= U(v_i) - U(v'_i) + u(s^*(v'_i), v'_i) - u(s^*(v'_i), v_i) = \int_{v'_i}^{v_i} u'_{v_i}(s^*(v), v) dv + \int_{v_i}^{v'_i} u'_{v_i}(s^*(v'_i), v) dv \\ &= \int_{v'_i}^{v_i} \left(F(v)^{|I|-1} - F(v'_i)^{|I|-1} \right) dv. \end{aligned}$$

If $v_i > v'_i$, then $F(v) \geq F(v'_i)$ for any $v \in [v'_i, v_i] \implies U(v_i) - u(a_i, v_i) \geq 0$.

1st-Price Auction

Solving for a symmetric PS-BNE $(s^*)_{j \in I}$

$$s^*(v_i) = v_i - \int_{\underline{v}}^{v_i} \left(\frac{F(v)}{F(v_i)} \right)^{|I|-1} dv$$

WT check s^* is optimal.

Take any $v_i \in V_i$ and $a_i \geq 0$.

Claim: Given others play s^* , $s^*(v_i)$ does weakly better than $a_i \forall a_i \in [s^*(\underline{v}), s^*(\bar{v})]$.

NB: s^* continuous and strictly increasing $\implies \exists! v'_i : a_i = s^*(v'_i) \forall a_i \in [s^*(\underline{v}), s^*(\bar{v})]$.

If $v_i < v'_i$, then

$$\begin{aligned} U_i(v_i) - u_i(a_i, v_i) &= U_i(v_i) - U_i(v'_i) + U_i(v'_i) - u_i(s^*(v'_i), v_i) \\ &= \int_{v'_i}^{v_i} \left(F(v)^{|I|-1} - F(v'_i)^{|I|-1} \right) dv \\ &= \int_{v_i}^{v'_i} \left(F(v'_i)^{|I|-1} - F(v)^{|I|-1} \right) dv \end{aligned}$$

and $F(v'_i) \geq F(v)$ for any $v \in [v_i, v'_i] \implies U(v_i) - u(a_i, v_i) \geq 0$.

Revenue Equivalence

Revenue Equivalence Theorem: Any auction setting such that

- (i) bidders' types are their valuation, drawn independently from compact convex set,
- (ii) the object is allocated to the bidder with the highest valuation,
- (iii) a bidder with the lowest possible valuation ($\underline{v}$) gets 0 in expected payoff in equilibrium

generates the same expected revenue to the auctioneer as the 2PA.

Revenue Equivalence

$v^{k:n}$: k -th highest valuation out of I bidders.

$\implies$ Revenue in 2PA: $V^{2:I}$.

Bids in 1PA:

$$\begin{aligned} s^*(v) &= v - \int_{\underline{v}}^v \left(\frac{F(s)}{F(v)} \right)^{I-1} ds = \frac{1}{F^{1:I-1}(v)} \left[F^{1:I-1}(v)v - \int_{\underline{v}}^v F^{1:I-1}(s) ds \right] \\ &= \frac{1}{F^{1:I-1}(v)} \int_{\underline{v}}^v s dF^{1:I-1}(s) \quad (\text{Integration by parts}) \\ &= \mathbb{E}[V^{1:I-1} | V^{1:I-1} < v] \end{aligned}$$

$\implies$ Revenue in 1PA:

$$s^*(V^{1:I}) = \mathbb{E}[V^{1:I-1} | V^{1:I-1} < V^{1:I}]$$

Revenue Equivalence: $\mathbb{E}[V^{2:I}] = \mathbb{E}[\mathbb{E}[V^{1:I-1} | V^{1:I-1} < V^{1:I}]]$

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Purification Theorem

MSNE hard to justify: although player is indifferent, they need to randomise in very particular way to make opponents indifferent as well.

Purification: Harsanyi (1973) provided a justification for MSNE of a normal-form game $\Gamma = \langle I, S, u \rangle$ as a limit case of perturbed games.

Suppose true preference is unobserved by opponents (random) and given by

$$\tilde{u}_i(s, \theta_i) := u_i(s) + \epsilon \theta_i^s$$

where θ_i^s are independent across players and drawn from a distribution F_i with density f_i .

Theorem

Fix a finite set of players I and strategy spaces S_i . For almost all payoff vectors $u = (u_i)_{i \in I}$ and for all independent and twice-differentiable densities f_i on $[-1, 1]^{|S_i|}$, any mixed strategy Nash equilibrium of the normal-form game $\Gamma = \langle I, S, u \rangle$ is the limit of a sequence of pure strategy Bayesian Nash equilibria of the Bayesian game with perturbed payoffs $(\tilde{u}_i)_{i \in I}$.

Note the limits of the result: “for *almost* all payoff vectors”

Higher-Order Beliefs

Coordination Game (bank runs, currency attacks)

		Col Player	
		Invest	Not Invest
Row Player	Invest	θ, θ	$\theta - 1, 0$
	Not Invest	$0, \theta - 1$	$0, 0$

Complete Information. NE?

$\theta < 0$: Not invest is strictly dominant and (NI,NI) the unique NE.

$\theta > 1$: Invest is strictly dominant and (I,I) the unique NE.

$\theta \in [0, 1]$: (NI,NI), (I,I), and mixed is NE.

Higher-Order Beliefs

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		Col Player	
		Invest	Not Invest
Row Player	Invest	θ, θ	$\theta - 1, 0$
	Not Invest	$0, \theta - 1$	$0, 0$

Incomplete Information

Suppose both players observe a signal about the state θ .

$$\theta_i := \theta + \varepsilon_i, \varepsilon_i \sim N(0, \sigma^2) \text{ iid.}$$

$$\theta \mid \theta_i \sim N(\theta_i, \sigma^2), \text{ because } \theta = \theta_i - \varepsilon_i$$

$$\theta_j \mid \theta_i := \theta \mid \theta_i + \varepsilon_j \mid \theta_i = \theta \mid \theta_i + \varepsilon_j \sim N(\theta_i, 2\sigma^2).$$

Implicitly, this is saying that players have uninformative or improper prior on θ that is uniform over the real line.

Why improper? because there is no uniform distribution over the real line; it cannot add-up to one if it has a constant pdf.

Higher-Order Beliefs

Incomplete Information

$\theta_i \mid \theta_j \sim N(\theta_i, \sigma^2)$ and $\theta_j \mid \theta_i \sim N(\theta_i, 2\sigma^2)$.

Claim: $s_i(\theta_j) := 1\{\theta_j > 1/2\}$ is an equilibrium.

- Given $s_j = 1\{\theta_j > 1/2\}$, player i 's payoff to investing conditional on θ_i and s_j is

$$\theta_i - \mathbb{P}(\theta_j \leq 1/2 \mid \theta_i) = \theta_i - \Phi\left(\frac{1/2 - \theta_i}{\sqrt{2}\sigma}\right)$$

strictly increasing in θ_i and zero when $\theta_i = 1/2$.

- $s_i(\theta_j) := 1\{\theta_j > 1/2\}$ is the unique best response to s_j .

Higher-Order Beliefs

Incomplete Information

$\theta \mid \theta_i \sim N(\theta_i, \sigma^2)$ and $\theta_j \mid \theta_i \sim N(\theta_i, 2\sigma^2)$.

WTS **Proposition**: In any eqm, $s_i(\theta_i) = 1$ a.e. on $(1/2, \infty)$ and $s_i(\theta_i) = 0$ a.e. on $(-\infty, 1/2)$.

Preliminaries: Define $f(\theta_i, \tilde{\theta}) := \theta_i - \Phi\left(\frac{\tilde{\theta} - \theta_i}{\sqrt{2}\sigma}\right)$.

Note: $\mathbb{P}(\theta_j < \tilde{\theta} \mid \theta_i) = \mathbb{P}\left(\frac{\theta_j - \theta_i}{\sqrt{2}\sigma} < \frac{\tilde{\theta} - \theta_i}{\sqrt{2}\sigma} \mid \theta_i\right) = \Phi\left(\frac{\tilde{\theta} - \theta_i}{\sqrt{2}\sigma}\right)$.

$f(\theta_i, \tilde{\theta})$ is continuous in $(\theta_i, \tilde{\theta})$, strictly increasing in θ_i , and strictly decreasing in $\tilde{\theta}$.

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Claim: $\forall \theta_i > 1, \mathbb{E}[u_i(l, s_j(\theta_j), \theta) \mid \theta_i] = \theta_i - \mathbb{E}[s_j(\theta_j) \mid \theta_i] > 0$.

- Note that: $\forall \delta > 0, f(\theta_i, \tilde{\theta}) > \delta \forall \theta_i \geq 1 + \delta$ and $\forall \tilde{\theta}$.
- Then $\forall \theta_i > 1, \mathbb{E}[u_i(l, s_j(\theta_j), \theta) \mid \theta_i] = \theta_i - \mathbb{E}[s_j(\theta_j) \mid \theta_i] \geq \theta_i = f(\theta_i, -\infty) > 0$
(where $f(\theta_i, -\infty) := \lim_{\tilde{\theta} \rightarrow -\infty} f(\theta_i, \tilde{\theta})$).

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For $k = 1, 2, \dots$, define $\bar{\theta}^{k+1} := \inf\{\theta_i | f(\theta_i, \bar{\theta}^k) > 0\}$, where $\bar{\theta}^1 = 1$.

Claim: $\bar{\theta}^k > \bar{\theta}^{k+1} \forall k$.

- True for $k = 0$.
- Induction: $\bar{\theta}^{k+1} < \bar{\theta}^k \implies 0 = f(\bar{\theta}^{k+2}, \bar{\theta}^{k+1}) = f(\bar{\theta}^{k+1}, \bar{\theta}^k) < f(\bar{\theta}^{k+1}, \bar{\theta}^{k+1})$
 $\therefore f$ strictly decreasing in 2nd argument and $\bar{\theta}^{k+1} < \bar{\theta}^k$.
- $f(\bar{\theta}^{k+2}, \bar{\theta}^{k+1}) < f(\bar{\theta}^{k+1}, \bar{\theta}^{k+1}) \implies \bar{\theta}^{k+2} < \bar{\theta}^{k+1} \therefore f$ strictly increasing in 1st argument.

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Claim: $\forall \theta_j > 1, \mathbb{E}[u_i(l, s_j(\theta_j), \theta) \mid \theta_j] = \theta_j - \mathbb{E}[s_j(\theta_j) \mid \theta_j] > 0$.

For $k = 1, 2, \dots$, define $\bar{\theta}^{k+1} := \inf\{\theta_j \mid f(\theta_j, \bar{\theta}^k) > 0\}$, where $\bar{\theta}^1 = 1$.

Claim: $\bar{\theta}^k > \bar{\theta}^{k+1} \forall k$.

Claim: (Induction step) If $s_j(\theta_j) = 1 \forall \theta_j > \bar{\theta}^k$, then $\forall \theta_j > \bar{\theta}^{k+1} \mathbb{E}[u_i(l, s_j(\theta_j), \theta) \mid \theta_j] > 0$.

- $\forall \theta_j > \bar{\theta}^{k+1}, \mathbb{E}[u_i(l, s_j(\theta_j), \theta) \mid \theta_j] \geq f(\theta_j, \bar{\theta}^k) > 0 = f(\bar{\theta}^{k+1}, \bar{\theta}^k)$
∴ f is strictly increasing in 1st arg.

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For $k = 1, 2, \dots$, define $\bar{\theta}^{k+1} := \inf\{\theta_i \mid f(\theta_i, \bar{\theta}^k) > 0\}$, where $\bar{\theta}^1 = 1$.

Claim: $\bar{\theta}^k > \bar{\theta}^{k+1} \forall k$.

Claim: (Induction step) If $s_j(\theta_j) = 1 \forall \theta_j > \bar{\theta}^k$, then $\forall \theta_i > \bar{\theta}^{k+1} \mathbb{E}[u_i(l, s_j(\theta_j), \theta) \mid \theta_i] > 0$.

Claim: At any BNE s , $s_i(\theta_i) = 1$ a.e. on $\theta_i > \bar{\theta}^k$, for all k and $i = 1, 2$.

- True for $k = 1$, $\mathbb{E}[u_i(l, s_j(\theta_j), \theta) \mid \theta_i] \geq \theta_i - 1 > 0$ for any $\theta_i > \bar{\theta}^1 = 1$, no matter s_j .
- Then, for any $s'_i : s'_i(\theta_i) \neq 1$ for a positive measure of $\theta_i > \bar{\theta}^1$ is strictly dominated by s_i s.t. $s_i = s'_i$ on $(-\infty, \bar{\theta}^1]$ and $s_i = 1$ on $(\bar{\theta}^1, \infty)$.
- Iterating the argument, for any k , for any $s'_i : s'_i(\theta_i) \neq 1$ for a positive measure of $\theta_i > \bar{\theta}^k$ is iteratedly strictly dominated by s_i s.t. $s_i = s'_i$ on $(-\infty, \bar{\theta}^k]$ and $s_i = 1$ on $(\bar{\theta}^k, \infty)$.

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Claim: $\forall \theta_i > 1, \mathbb{E}[u_i(l, s_j(\theta_j), \theta) \mid \theta_i] = \theta_i - \mathbb{E}[s_j(\theta_j) \mid \theta_i] > 0$.

For $k = 1, 2, \dots$, define $\bar{\theta}^{k+1} := \inf\{\theta_i \mid f(\theta_i, \bar{\theta}^k) > 0\}$, where $\bar{\theta}^1 = 1$.

Claim: $\bar{\theta}^k > \bar{\theta}^{k+1} \forall k$.

Claim: (Induction step) If $s_j(\theta_j) = 1 \forall \theta_j > \bar{\theta}^k$, then $\forall \theta_i > \bar{\theta}^{k+1} \mathbb{E}[u_i(l, s_j(\theta_j), \theta) \mid \theta_i] > 0$.

Claim: At any BNE s , $s_i(\theta_i) = 1$ a.e. on $\theta_i > \bar{\theta}^k$, for all k and $i = 1, 2$.

Claim: $\bar{\theta}^k > 0 \forall k$.

- Note that: $\forall \delta > 0, f(\theta_i, \tilde{\theta}) < -\delta \forall \theta_i \leq -\delta$ and $\forall \tilde{\theta}$.
- Then $\forall \theta_i < 0, \mathbb{E}[u_i(l, s_j(\theta_j), \theta) \mid \theta_i] = \theta_i - \mathbb{E}[s_j(\theta_j) \mid \theta_i] \leq \theta_i = f(\theta_i, \infty) < 0$
(where $f(\theta_i, \infty) := \lim_{\tilde{\theta} \rightarrow \infty} f(\theta_i, \tilde{\theta})$).
- Then, as $f(\bar{\theta}^k, \bar{\theta}^k) > 0 > f(0, \bar{\theta}^k)$ and f is strictly increasing in 1st argument, then $\bar{\theta}^k > 0 \forall k$.

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Claim: $\forall \theta_i > 1, \mathbb{E}[u_i(l, s_j(\theta_j), \theta) \mid \theta_i] = \theta_i - \mathbb{E}[s_j(\theta_j) \mid \theta_i] > 0$.

For $k = 1, 2, \dots$, define $\bar{\theta}^{k+1} := \inf\{\theta_i \mid f(\theta_i, \bar{\theta}^k) > 0\}$, where $\bar{\theta}^1 = 1$.

Claim: $\bar{\theta}^k > \bar{\theta}^{k+1} \forall k$.

Claim: (Induction step) If $s_j(\theta_j) = 1 \forall \theta_j > \bar{\theta}^k$, then $\forall \theta_i > \bar{\theta}^{k+1} \mathbb{E}[u_i(l, s_j(\theta_j), \theta) \mid \theta_i] > 0$.

Claim: At any BNE s , $s_i(\theta_i) = 1$ a.e. on $\theta_i > \bar{\theta}^k$, for all k and $i = 1, 2$.

Claim: $\bar{\theta}^k > 0 \forall k$.

Claim: $\lim_{k \rightarrow \infty} \bar{\theta}^k = 1/2$.

- $\{\bar{\theta}^k\}_k$ decreasing sequence, bounded below by 0 $\implies$ it converges to some $\bar{\theta}^\infty \geq 0$, by monotone convergence theorem.
- $0 = \lim_{k \rightarrow \infty} f(\bar{\theta}^{k+1}, \bar{\theta}^k) = f(\bar{\theta}^\infty, \bar{\theta}^\infty) = \bar{\theta}^\infty - \Phi\left(\frac{\bar{\theta}^\infty - \bar{\theta}^\infty}{\sqrt{2}\sigma}\right) = \bar{\theta}^\infty - 1/2$.

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Preliminaries: Define $f(\theta_i, \tilde{\theta}) := \theta_i - \Phi\left(\frac{\tilde{\theta} - \theta_i}{\sqrt{2}\sigma}\right)$.

Fully symmetric arguments:

Claim: $\forall \theta_i < 0, \mathbb{E}[u_i(l, s_j(\theta_j), \theta) | \theta_i] = \theta_i - \mathbb{E}[s_j(\theta_j) | \theta_i] < 0$.

For $k = 1, 2, \dots$, define $\underline{\theta}^{k+1} := \sup\{\theta_j | f(\theta_j, \underline{\theta}^k) < 0\}$, where $\underline{\theta}^1 = 0$.

Claim: $\underline{\theta}^k < \underline{\theta}^{k+1} \forall k$.

Claim: (Induction step) If $s_j(\theta_j) = 0 \forall \theta_j < \underline{\theta}^k$, then $\forall \theta_i < \underline{\theta}^{k+1} \mathbb{E}[u_i(l, s_j(\theta_j), \theta) | \theta_i] < 0$.

Claim: At any BNE s , $s_i(\theta_i) = 0$ a.e. on $\theta_i < \underline{\theta}^k$, for all k and $i = 1, 2$.

Claim: $\underline{\theta}^k < 0 \forall k$.

Claim: $\lim_{k \rightarrow \infty} \underline{\theta}^k = 1/2$.

Incomplete Information

$\theta \mid \theta_i \sim N(\theta_i, \sigma^2)$ and $\theta_j \mid \theta_i \sim N(\theta_i, 2\sigma^2)$.

Proposition: In any eqm, $s_i(\theta_i) = 1$ a.e. on $(1/2, \infty)$ and $s_i(\theta_i) = 0$ a.e. on $(-\infty, 1/2)$.

NB: proposition holds $\forall \sigma$. Taking $\sigma \downarrow 0$ selects unique NE.

Global game approach to selection of NE.

Higher-Order Beliefs

Why higher-order beliefs?

Player i will invest if θ_i is high enough, regardless of whether j invests.

Player i knows player j will also invest if θ_j is high enough.

That makes Player i more amenable to investing at lower threshold for θ_i , as j will invest regardless of what i does for high enough θ_j .

Iterating the argument on players' beliefs has higher-order beliefs working in the background to refine what the opponent will do.

Common knowledge of rationality is doing all the heavy-lifting in determining how players behave!

Overview

1. Motivation
2. Bayesian Games
3. Bayesian Nash Equilibrium
4. Auctions
5. Purification Theorem
6. Higher-Order Beliefs
7. More

The Originals: Harsanyi (1967 MnSc, 1968 MnSc, 1973 IJGT).

Auctions: Milgrom (1981 Ecta, 2003), Myerson (1981 MOR), Athey & Haile (2002 Ecta).
Auctions with budgets: e.g., Ghosh (2021 GEB).

Global Games: Morris & Shin (1998 AER, 2002 AER).

Other Topics: Voting (Feddersen & Pesendorfer, 1997 Ecta), **Media Bias** (Gentzkow & Shapiro, 2006 JPE).

No-Trade Theorem: Milgrom & Stokey (1982 JET).

Experiments: Winner's curse Charness & Levin (2009 AEJMicro), Overbidding and QRE: Goeree, Holt, & Palfrey (2002 JET); Camerer, Nunnari, & Palfrey (2016 GEB); and Charness, Levin, & Schmeidler (2019 JET).