11. Nash Equilibrium

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Overview

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Please predict choice frequencies.

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Now: Sharpening prediction, bridging the disconnect.

Goal: Nash equilibrium, GT's gold standard.

Used *everywhere*. (Not just economics or even social sciences.)

Overview

1. Motivation
2. Nash Equilibrium
3. Examples
4. Normal-Form Refinements and Generalizations of Nash Equilibrium
5. More

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$s \in S$ is a **pure strategy Nash equilibrium** iff $\forall i$, $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$ $\forall s_i' \in S_i$ (s_i is BR to s_{-i}).

Note the difference: equilibrium, equilibrium payoff, equilibrium outcome.

Pure strategy $\equiv$ degenerate mixed strategies.

Non-degenerate mixed strategies $\not\equiv$ totally mixed strategies.

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$\sigma \in \Sigma$ is a **Nash equilibrium** if $\forall i$, $u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i})$ $\forall \sigma_i' \in \Sigma_i$ (σ_i is BR to σ_{-i}).

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Player *i*'s best-response correspondence is given by $b_i : \Sigma_{-i} \rightrightarrows \Sigma_i$ s.t. $b_i(\boldsymbol{\sigma}_{-i}) := \arg\max_{\boldsymbol{\sigma}_i \in \Sigma_i} u_i(\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_{-i})$.

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Remark

$\boldsymbol{\sigma}$ is a Nash equilibrium iff $\boldsymbol{\sigma} \in b(\boldsymbol{\sigma})$.

Nash Equilibrium: Interpretation

(1) **Resulting from introspection**.

Everyone is BR to everyone else. If not, they'd prefer to do something else.

NE is rationalisable. All rationalisable strategy profiles are NE?

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Generally: What do to when there are multiple equilibria?
How does one decide which one is to be played?

Nash Equilibrium: Interpretation

(1) **Resulting from introspection**.

(2) **An outcome of learning, a steady state of a long-run adjustment process.**

Can help in selection of an equilibrium.

Fudenberg & Levine (1998); see also Fudenberg & Levine (2016) and Fudenberg (2022) for surveys.

Learning and dynamic adjustment: in next year's theory topics course!

Growing literature on estimating equilibria in games; what about dynamic adjust toward equilibrium? (workshop on GT & metrics sponsored by cemmap)

Existence of a Nash Equilibrium

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  - Write down assumptions formally, but there is no equilibrium; your model is unable to make predictions!
  - Not a good model of the agents' behaviour.
- Existence results ensure that, under a given set of assumptions, your model *works* (whether it makes good predictions or not it is another matter)

Existence of a Nash Equilibrium

Kakutani's Fixed-Point Theorem

Let $X \subset \mathbb{R}^n$ be nonempty, compact, and convex. If $F : X \rightrightarrows X$ is nonempty-valued, compact-valued, convex-valued, and uhc, then $\exists x \in X : x \in F(x)$, i.e., there is a fixed point of F.

Existence of a Nash Equilibrium

Theorem

Let $\Gamma = \langle I, S, u \rangle$ be a normal-form game s.t. $|I| < \infty$, and, $\forall i \in I$, S_i is a nonempty, compact, and convex subset of $\mathbb{R}^n$. If $u_i : S \to \mathbb{R}$ is continuous in S and quasiconcave in s_i, then, there is a PSNE.

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Proof

Let $b : S \rightrightarrows S$ be s.t. $b(s) = \times_{i \in I} b_i(s_{-i})$, where $b_i(s_{-i}) := \arg\max_{s_i \in S_i} u_i(s_i, s_{-i})$.

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(i) b is nonempty-, compact-valued, and UHC.

u_i continuous, S_i is compact $\implies$ b_i nonempty-value, compact-valued and UHC $\forall i \in I$ (by Berge's maximum theorem).

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(ii) b is convex-valued.

u_i is quasiconcave in s_i $\implies$ $b_i(s_{-i})$ is convex $\forall s_{-i} \in S_{-i}, \forall i \in I$ $\implies$ b convex-valued.

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By Kakutani's fixed-point theorem, $\exists s \in b(s)$. □

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Corollary

Let $\Gamma = \langle I, S, u \rangle$ be a normal-form game s.t. $|I|, |S| < \infty$. Then, there is a NE, possibly in mixed-strategies.

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Proof

- Game in mixed-strategies as a different game, $\tilde{\Gamma} = \langle I, \Sigma, \tilde{u} \rangle$, with $\tilde{u}_i(\sigma) = \mathbb{E}_{\sigma}[u_i]$.

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- Σ_i as a nonempty, compact, and convex subset of $[0, 1]^{|S_i|}$
- $\tilde{u}_i$ continuous in σ and linear (hence quasiconcave) in σ_i.
- Conditions of theorem met, hence $\exists$ PSNE σ of $\tilde{\Gamma}$, which is NE (possibly mixed) of Γ. $\square$

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Nash provided a different proof, based on Brouwer's fixed point theorem:

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Let X be a nonempty, compact, and convex subset of $\mathbb{R}^n$. If $f : X \to X$ is continuous, then f admits a fixed-point $x = f(x)$.

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Instructive proof by Geneakoplos.

Theorem

Let $\Gamma = \langle I, S, u \rangle$ be a normal-form game s.t. $|I| < \infty$, and, $\forall i \in I$, S_i is a nonempty, compact, and convex subset of $\mathbb{R}^n$. If $u_i : S \to \mathbb{R}$ is continuous in S and concave in s_i, then there is a PSNE.

Existence of a Nash Equilibrium

Proof

- Let $\phi_i : S \to S_i$ be s.t. $\phi_i(s) := \arg\max_{s_i' \in S_i} u_i(s_i', s_{-i}) - ||s_i - s_i'||^2$.

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- $u_i(s'_i, s_{-i})$ is concave in s'_i and $-||s_i - s'_i||^2$ is strictly concave $\implies u_i(s'_i, s_{-i}) - ||s_i - s'_i||^2$ strictly concave and continuous in s'_i.

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- Let $\phi(s) := (\phi_i(s))_{i \in I}$. $\phi : S \rightrightarrows S$ continuous, $S \subset \mathbb{R}^n$ convex, compact $\implies \phi$ has fixed point $s = \phi(s)$ (by Brouwer's fixed-point theorem).

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  - u_i concave $\implies u_i(\alpha s_i' + (1-\alpha)s_i, s_{-i}) - u_i(s) \geq \alpha(u_i(s_i', s_{-i}) - u_i(s))\ \forall \alpha \in (0,1)$.
  - NB: $||s_i - (\alpha s_i' + (1-\alpha)s_i)||^2 = \alpha^2 ||s_i - s_i'||^2$.

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  - As $\phi_i(s) = s_i \implies \forall \alpha \in (0,1)$

$$0 \geq u_i((\alpha s_i' + (1-\alpha)s_i), s_{-i}) - ||s_i - (\alpha s_i' + (1-\alpha)s_i)||^2$$
$$- \left(\max_{s_i'' \in S_i} u_i(s_i'', s_{-i}) - ||s_i - s_i''||^2 \right)$$

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$$
\begin{aligned}
0 \geq & u_i((\alpha s_i' + (1-\alpha)s_i), s_{-i}) - ||s_i - (\alpha s_i' + (1-\alpha)s_i)||^2 \\
& - \left(\max_{s_i'' \in S_i} u_i(s_i'', s_{-i}) - ||s_i - s_i''||^2 \right) \\
= & u_i((\alpha s_i' + (1-\alpha)s_i), s_{-i}) - ||s_i - (\alpha s_i' + (1-\alpha)s_i)||^2 - u_i(s)
\end{aligned}
$$

Existence of a Nash Equilibrium

Proof

- $\phi_i(s) := \arg\max_{s_i' \in S_i} u_i(s_i', s_{-i}) - ||s_i - s_i'||^2$. $\phi(s) := (\phi_i(s))_{i \in I}$. $\exists s \in S : s = \phi(s)$.
- WTS $s_i \in \arg\max_{s_i \in S_i} u_i(s_i, s_{-i})$.
  - Suppose not, i.e., $\exists s_i' \in S_i : u_i(s_i', s_{-i}) > u_i(s)$ for some i.
  - u_i concave $\implies u_i(\alpha s_i' + (1-\alpha)s_i, s_{-i}) - u_i(s) \geq \alpha(u_i(s_i', s_{-i}) - u_i(s))\ \forall \alpha \in (0,1)$.
  - NB: $||s_i - (\alpha s_i' + (1-\alpha)s_i)||^2 = \alpha^2 ||s_i - s_i'||^2$.
  - As $\phi_i(s) = s_i \implies \forall \alpha \in (0,1)$

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\begin{aligned}
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\geq & \alpha(u_i(s_i', s_{-i}) - u_i(s)) - \alpha^2 ||s_i - s_i'||^2
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  - Choosing α s.t. $\alpha < (u_i(s'_i, s_{-i}) - u_i(s))/||s_i - s'_i||^2$ delivers
    $0 \geq \alpha(u_i(s'_i, s_{-i}) - u_i(s)) - \alpha^2 ||s_i - s'_i||^2 > \alpha^2 - \alpha^2 = 0$. a contradiction. □

Symmetric Nash Equilibria

Definition

Let $\Gamma = \langle I, S, u\rangle$ be a normal-form game. Γ is **symmetric** iff $\forall i, j \in I$, $S_j = S_i$, and $u_i(s_i, s_{-i}) = u_j(s_j, s_{-j})$ for $s_i = s_j$ and $s_{-i} = s_{-j}$.

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Let $\Gamma = \langle I, S, u \rangle$ be a symmetric normal-form game s.t. $|I| < \infty$, and, $\forall i \in I$, S_i is a nonempty, compact, and convex subset of $\mathbb{R}^n$. If $u_i : S \to \mathbb{R}$ is continuous in S and quasiconcave in s_i, then there is a PSNE s s.t. $s_i = s_j \in S_i \forall i, j \in I$.

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Proof

Define $b : S_i \rightrightarrows S_i$ s.t. $\tilde{b}(s_i) = b_i(s_{-i})$ for $s_{-i} = (s_i)_{j \in -i}$; b_i is player i's best-response correspondence.

$\tilde{b}$ nonempty-, compact-, and convex-valued, and UHC $\implies$ Kakutani's fixed-point theorem applies. $\square$

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Let X be a nonempty, compact, convex subset of a locally convex Hausdorff (e.g., vector) space and that $f : X \rightrightarrows X$ is nonempty- and convex-value correspondence with a closed graph. Then $\exists x \in X : x \in f(x)$.

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Does 2PA have continuous payoffs? What to do then? See Reny (1999 Ecta) "On the Existence of Pure and Mixed Strategy Nash Equilibria in Discontinuous Games"

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(iv) However... weakly dominated strategies *can* be played with positive probability at a Nash equilibrium.

Relation to Dominance

		Col Player A	Col Player B
Row Player	A	1,1	0,0
	B	0,0	0,0

NE: (A,A) and (B,B)

Characterising Equilibria

		Col Player A	Col Player B
Row Player	A	1,1	2,3
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PSNE: (A,B) and (B,A)

Characterising Equilibria

Remark

$\boldsymbol{\sigma}$ is a Nash equilibrium if and only if $\forall i \in I$, (i) $u_i(s_i, \boldsymbol{\sigma}_{-i}) \geq u_i(s_i', \boldsymbol{\sigma}_{-i})\ \forall s_i \in \text{supp}(\boldsymbol{\sigma}_i), s_i' \in S_i$, and (ii) $u_i(s_i, \boldsymbol{\sigma}_{-i}) = u_i(s_i', \boldsymbol{\sigma}_{-i})\ \forall s_i, s_i' \in \text{supp}(\boldsymbol{\sigma}_i)$.

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Best response condition: any pure strategy in the support must be best response to $\boldsymbol{\sigma}_{-i}$, $u_i(s_i, \boldsymbol{\sigma}_{-i}) \geq u_i(s_i', \boldsymbol{\sigma}_{-i})$ for any $s_i \in \text{supp}(\boldsymbol{\sigma}_i)$ and $s_i' \in S_i$.

MSNE indifference condition: must get same payoff $u_i(s_i, \boldsymbol{\sigma}_{-i}) = u_i(s_i', \boldsymbol{\sigma}_{-i})$ for any pure strategy in the support, $s_i, s_i' \in \text{supp}(\boldsymbol{\sigma}_i)$.

Characterising Equilibria

		Col Player	
		A	B
Row Player	A	1,1	2,3
	B	3,2	0,0

PSNE: (A,B) and (B,A)

Given σ_C,

$$u_R(A, \sigma_C) \geq u_R(B, \sigma_C) \implies \sigma_C(A)1 + (1 - \sigma_C(A))2 \geq \sigma_C(A)3 + (1 - \sigma_C(A))0$$

$$\sigma_C(A) \leq \frac{1}{2}.$$

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$$\sigma_C(A) \leq \frac{1}{2}.$$

NE: (A,B), (B,A), and $(\sigma_R, \sigma_C) : \sigma_R(A) = \sigma_C(A) = 1/2$.

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Players don't willingly randomize; they are *indifferent*. How to interpret MSNE?

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As stochastic choice: players look like they are randomizing, but could be random utility players ('purification' e.g. Harsanyi 1973 IJGT – more later), unobserved information acquisition (Gonçalves 2024 WP), etc.

MSNE *can be* rationalized as the limit outcome of one such situation.

Robustness

Will limit of equilibria be an equilibrium of the limit game? (is set of NE UHC?)

Robustness

Proposition

Let $S := \times_{i \in I} S_i$ be such that S_i is nonempty, compact, and convex subset of $\mathbb{R}^{n_i}$, $T \subseteq \mathbb{R}^m$, $u_i : S \times T \to \mathbb{R}$. Let $S^{NE}(t) := \left\{ s \in S \mid s_i \in \arg\max_{s_i' \in S_i} u_i(s_i', s_{-i}, t) \right\}$ be set of NE of the game $\Gamma_t := \langle I, S, u^t \rangle$, where $u^t := (u_i(\cdot, t))_{i \in I}$.

If (i) u_i is continuous in (s, t), and (ii) $S^{NE}(t')$ is nonempty for any t' in a neighborhood of t, then S^{NE} is UHC at t.

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Proof

- Take $(s^n, t^n) \to (s, t)$, where $s^n \in S^{NE}(t^n)$ for all n.

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- Then

$$u_i(s_i, s_{-i}, t) = \lim_{n \to \infty} u_i(s_i^n, s_{-i}^n, t^n) = \lim_{n \to \infty} \max_{s_i' \in S_i} u_i(s_i', s_{-i}^n, t^n) = \max_{s_i' \in S_i} u_i(s_i', s_{-i}, t).$$

- Hence, $s \in S^{NE}(t)$ and S^{NE} is uhc (and compact-valued) at t. □

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Issue: not all NE are robust; small mistakes may 'kill fragile equilibria'. More later.

Overview

Common Value All-Pay Auction

I bidders, all value object at $v > 0$. Bids $s_i \geq 0$.

Payoffs: always pay bid; win if bid highest; ties broken uniformly at random.

$$u_i(s_i, s_{-i}) = \mathbf{1}\{s_i = \max_j s_j\} \cdot \frac{1}{|\{j \in I \mid s_j = \max_\ell s_\ell\}|} v - s_i$$

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Claim 1: No one bids above v: strictly dominated.

Claim 2: No PSNE in this game.

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(b) $i = \arg\max_j s_j$, then i wants to deviate to $s_i' = s_i - \varepsilon$ for small enough $\varepsilon > 0$.

(c) If $\max_j s_j = v$, $i, \ell \in \arg\max_j s_j$ with $i \neq \ell$, then i wants to deviate to $s_i' = 0$.

Common Value All-Pay Auction

Is there NE?

Common Value All-Pay Auction

Is there NE? Let's try to construct a symmetric MSNE $\sigma_i = \sigma_j$ (CDF):

- NB: $\sigma_i \in \Delta([0, v])$; (why?)

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- NB: $\sigma_i \in \Delta([0, v])$; (why?)

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- As $\sigma_j(v) = 1 \iff \overline{u} = 0$.
- No atoms: $\sigma_j(s_j) = 0 \implies s_j = 0$.
- By construction, $\forall s_i \in [0, v]$, $u_i(s_i, \sigma_{-i}) = \overline{u} = 0$. (Indifference).

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- By construction, $\forall s_i \in [0, v]$, $u_i(s_i, \sigma_{-i}) = \bar{u} = 0$. (Indifference).
- Conclusion: Competition left bidders with 0 expected surplus from the auction.

Model of Sales (Varian 1980 AER)

Two firms sell a good to a unit mass of consumers with reservation price of £1 (WTP).

Firms set prices simultaneously.

Each firm has loyal consumers (insofar as price doesn't exceed reservation price):

- 1/4 of consumers always buy from firm 1 and 1/4 from firm 2.
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(v) Pricing below $1/3$ yields $\frac{3}{4}p_i < \frac{1}{4}$.

(i.e., in expectation worse than pricing at $p_i \in [1/3, 1]$.)

Overview

A Silly Story

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The options are *Heretic* and *Megalopolis* and they'll watch the most voted movie.

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But there is a NE in which everyone votes to watch *Megalopolis* again!

This is very silly and we want to rule out silly predictions in our model.

Trembling-Hand Perfection

Idea: non-zero probabilities on each pure strategy capture the notion of unavoidable mistakes.

Define $\Delta_\varepsilon(S_i) := \{\sigma_i \in \Sigma_i \mid \sigma_i(s_i) \geq \varepsilon(s_i), \forall s_i \in S_i\}$ for $\varepsilon : \cup_{i \in I} S_i \to (0, 1)$.

$\Delta_\varepsilon(S_i)$: a restricted strategy space for player i, with fully mixed strategies.

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An ε-constrained Nash equilibrium of game $\Gamma = \langle I, S, u\rangle$ is a pure strategy Nash equilibrium of the perturbed game $\langle I, \times_i \Delta_\varepsilon(S_i), u\rangle$.

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Remark

For any ε, $\Delta_\varepsilon(S_i)$ is compact. Hence, insofar as $\Delta_\varepsilon(S_i)$ is nonempty for all i, there is an ε-constrained NE.

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Definition

A NE σ of game Γ is trembling-hand perfect if $\exists(\varepsilon^n)_n$ s.t. $\varepsilon^n : \cup_{i \in I} S_i \rightarrow (0,1)$ with $\varepsilon^n(s_i) \rightarrow 0 \ \forall s_i \in S_i$, and an associated sequence of ε^n-constrained NE $\sigma^n : \sigma^n \rightarrow \sigma$.

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Every finite game has a THPE.

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Take any sequence of $\boldsymbol{\varepsilon}^n$-constrained equilibrium for $n > N$; as it lives in a compact set, it admits a convergent subsequence.

Trembling-Hand Perfection

Definition

A NE $\boldsymbol{\sigma}$ of game Γ is trembling-hand perfect if $\exists(\boldsymbol{\varepsilon}^n)_n$ s.t. $\boldsymbol{\varepsilon}^n : \cup_{i\in I} S_i \to (\mathbf{0}, \mathbf{1})$ with $\boldsymbol{\varepsilon}^n(s_i) \to \mathbf{0}\ \forall s_i \in S_i$, and an associated sequence of $\boldsymbol{\varepsilon}^n$-constrained NE $\boldsymbol{\sigma}^n : \boldsymbol{\sigma}^n \to \boldsymbol{\sigma}$.

Theorem

Every finite game has a THPE.

Proof

Let $\tilde{\boldsymbol{\varepsilon}}^n := \max_{s_i \in \cup_{i\in I}} \boldsymbol{\varepsilon}^n(s_i)$.

By convergence of $\tilde{\boldsymbol{\varepsilon}}^n \to \mathbf{0}$, $\exists N : \forall n > N$, $\boldsymbol{\Delta}_{\boldsymbol{\varepsilon}^n}(S_i) \neq \emptyset \forall i \in I$.

Then, $\forall n > N\ \exists \boldsymbol{\varepsilon}^n$-constrained equilibrium.

Take any sequence of $\boldsymbol{\varepsilon}^n$-constrained equilibrium for $n > N$; as it lives in a compact set, it admits a convergent subsequence.

As $u_i : \boldsymbol{\Sigma} \to \mathbb{R}$ is continuous $\forall i$, subsequence converges to a NE of original game.

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Proposition

A NE $\boldsymbol{\sigma}$ of game Γ is THPE iff $\exists$ sequence of fully mixed strategy profiles $\boldsymbol{\sigma}^n \to \boldsymbol{\sigma}$: $\forall i$ and n, $\boldsymbol{\sigma}_i$ is a best response to $\boldsymbol{\sigma}^n_{-i}$.

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Proof: $\Longrightarrow$

If $\boldsymbol{\sigma}$ is THPE $\Longrightarrow$ $\exists \boldsymbol{\varepsilon}^n$: $\boldsymbol{\sigma}$ is limit of $\boldsymbol{\varepsilon}^n$-constrained NE $\boldsymbol{\sigma}^n$.

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WTS $\exists$ subsequence $\boldsymbol{\sigma}^m$ of $\boldsymbol{\sigma}^n$ s.t. $\boldsymbol{\sigma}_i$ is a best response to $\boldsymbol{\sigma}_{-i}^m$.

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- Suppose not. Then $\exists s_i \in \text{supp}(\boldsymbol{\sigma}_i)$ and some further subsequence $\boldsymbol{\sigma}^k$ s.t. $\forall k$, $u_i(s_i, \boldsymbol{\sigma}_{-i}^k) < u_i(s_i^k, \boldsymbol{\sigma}_{-i}^k)$ for some $s_i^k \in S_i$.

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- Let $\tilde{\boldsymbol{\varepsilon}}^n := \max_{s_i \in \cup_{i \in I}} \boldsymbol{\varepsilon}^n(s_i)$. We must have $\boldsymbol{\sigma}^k_i(s_i) \leq \tilde{\boldsymbol{\varepsilon}}^k$. Supposing otherwise yields:

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If σ is THPE $\implies$ $\exists \varepsilon^n$: σ is limit of ε^n-constrained NE σ^n.

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$$u_i(\sigma^k_i, \sigma^k_{-i}) = \sigma^k_i(s_i) u_i(s_i, \sigma^k_{-i}) + \sigma^k_i(s^k_i) u_i(s^k_i, \sigma^k_{-i}) + \sum_{s'_i \neq s_i, s^k_i} \sigma_i(s'_i) u_i(s'_i, \sigma^k_{-i})$$

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$$< \tilde{\varepsilon}^k u_i(s_i, \boldsymbol{\sigma}^k_{-i}) + (\boldsymbol{\sigma}^k_i(s^k_i) + \boldsymbol{\sigma}^k_i(s_i) - \boldsymbol{\varepsilon}^k) u_i(s^k_i, \boldsymbol{\sigma}^k_{-i}) + \sum_{s'_i \neq s_i, s^k_i} \boldsymbol{\sigma}_i(s'_i) u_i(s'_i, \boldsymbol{\sigma}^k_{-i}).$$

- But then $\boldsymbol{\sigma}^k_i(s_i) \to 0$, which contradicts the fact that $\boldsymbol{\sigma}^k_i(s_i) \to \boldsymbol{\sigma}_i(s_i) > 0$.

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Proof: $\Longleftarrow$

- Let $\boldsymbol{\varepsilon}^n(s_i) = \boldsymbol{\sigma}_i^n(s_i)$ if $\boldsymbol{\sigma}_i(s_i) = \mathbf{0}$ and $\boldsymbol{\varepsilon}^n(s_i) = \boldsymbol{\sigma}_i(s_i)/n$ if $\boldsymbol{\sigma}_i(s_i) > \mathbf{0}$.

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- Let $\overline{u}^n := u_i(s_i, \sigma_{-i}^n) \geq u_i(s_i', \sigma_{-i}^n)\ \forall s_i' \in S_i$. Let $A := \text{supp}(\sigma_i)$ and $B := S_i \setminus A$.
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and therefore σ_i^n is ε^n-constrained BR to σ_{-i}^n. (Feasible $\forall n > 2|S|$.) □

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Proposition

(i) If σ is fully mixed NE, then it is THPE.

Other notions: strong or strict equilibrium

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Correlated Equilibrium

And Now for Something Completely Different...

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Definition

A probability distribution $p \in \Delta(S)$ is a **correlated equilibrium** of a normal-form game $\Gamma = \langle I, S, u \rangle$ if $\forall i$ and $\forall s_i : p(s_i) > 0$

$$\sum_{s_{-i} \in S_{-i}} p(s_{-i}|s_i) u_i(s_i, s_{-i}) \geq \sum_{s_{-i} \in S_{-i}} p(s_{-i}|s_i) u_i(s_i', s_{-i}), \quad \forall s_i' \in S_i.$$

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Every Nash equilibrium is a correlated equilibrium

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$$\sum_{s_{-i} \in S_{-i}} p(s_{-i}|s_i) u_i(s_i, s_{-i}) \geq \sum_{s_{-i} \in S_{-i}} p(s_{-i}|s_i) u_i(s_i', s_{-i}), \quad \forall s_i' \in S_i.$$

p as correlated recommendations to the agents s.t. everyone wants to follow a the recommendations.

Proposition

Every Nash equilibrium is a correlated equilibrium

Corollary

CE exists in finite games

Correlated Equilibrium

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Convex hull of set of NE is subset of set of CE (also convex).

$\Longrightarrow$ Can attain any convex combination of NE payoffs. That's it?

Correlated Equilibrium: An Example

Coordination Game

		Col Player	
		A	B
Row Player	A	5,1	0,0
	B	4,4	1,5

- NE?

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- Suppose the players seek a mediator to help them. The mediator proposes the following:

  > I'm going to toss a die. If it turns up either 1 or 2, will tell the Row player to play A, and otherwise I will tell them to play B. If it turns up either 5 or 6, will tell the Column player to play B, and otherwise I will tell them to play A.

  Do the players want to follow the advice?

Correlated Equilibrium: An Example

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		Col Player	
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The mediator's proposal:

I'm going to toss a die. If it turns up either 1 or 2, will tell Row to play A, and otherwise I will tell them to play B. If it turns up either 5 or 6, will tell Column to play B, and otherwise I will tell them to play A.

- If Row is told to play A, they know die turned up $\{1, 2\}$. $\Longrightarrow$ they also know Column will be told to play A half the times and B the remainder.
- Expected payoff: $1/2 \cdot 5 + 1/2 \cdot 0$.

If they didn't follow the recommendation, then they'd get $1/2 \cdot 4 + 1/2 \cdot 1$; cannot do any better.

Correlated Equilibrium: An Example

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- If Row is told to play B, they know die turned up $\{3, 4, 5, 6\}$. $\Longrightarrow$ they also know Column will be told to play A half the times and B the remainder.
- Expected payoff: $1/2 \cdot 5 + 1/2 \cdot 0$.

  If they didn't follow the recommendation, then they'd get $1/2 \cdot 4 + 1/2 \cdot 1$; cannot do any better.
- Symmetric game: symmetric arguments apply for Column.
- Note: Row gets $1/3 \cdot [u_R(A, A) + u_R(B, A) + u_R(B, B)] = 1/310$; outside convex hull of NE payoffs.

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- If Row is told to play B, they know die turned up $\{3, 4, 5, 6\}$. $\implies$ they also know Column will be told to play A half the times and B the remainder.
- Expected payoff: $1/2 \cdot 5 + 1/2 \cdot 0$.

  If they didn't follow the recommendation, then they'd get $1/2 \cdot 4 + 1/2 \cdot 1$; cannot do any better.
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Moral of the story: correlated eqm allows you to do more!

Overview

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Level-*k*

WT incorporate reasoning mistakes.

Level-*k*

Cognitive Hierarchies

Endogenous Depth of Reasoning

Issues

(i) as if people have very unrealistic beliefs.

(ii) not well defined for arbitrary games.

(iii) “level” unstable even across dominance-solvable games.

(iv) individual’s reasoning seems to depend on payoffs: take “more steps” of IESDS the higher the stakes.

(v) individual’s reasoning seems to react to relative incentives smoothly.

Possible ways forward:

- pure stochastic choice as **Quantal Response Equilibrium** (McKelvey & Palfrey, 1995 GEB);
- model steps of reasoning via sampling **(Sequential) Sampling Equilibrium** (Osborne & Rubinstein 1998 AER, 2003 GEB; Gonçalves 2023 WP).