12. Incomplete Information

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Overview

Complete information assumption implies players know others' payoffs.

Examples:

- goal keeper may not really know how effortful it is for the penalty kicker to shoot right instead of left;
- firms may not know other firms' cost structure;
- voters may not know how other voters' preferences;
- consumers may be unsure of how much they value a good;
- investors may not know what is the value of an asset;
- firm may not know how productive a given job candidate is;
- a researcher may not know how difficult a problem they're working on is.

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Today's agenda: formalising games of incomplete information and examining applications.

Overview

1. Motivation
2. Bayesian Games
3. Bayesian Nash Equilibrium
4. Auctions
5. Purification Theorem
6. Higher-Order Beliefs
7. More

Overview

Representing Incomplete Information

Definition

A game is of **incomplete information** when at least one player does not know the payoff that some player receives from some strategy profile.

How to model uncertainty?

Harsanyi's modelling insight:

- Transform incomplete info game into complete info with Nature moving at start of game.
- Realisation of nature's actions determines players' payoffs.
- Assumption: CK of prob. distrib. used by Nature.

Players have a belief about others' preferences and there is common knowledge of such beliefs.

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A **Bayesian game** is a tuple $\langle I, A, u, \Theta, \rho \rangle$, where

(i) Players: I;

(ii) Player i's action space A_i; Space of action profiles $a \in A := \times_{i \in I} A_i$;

(iii) Player i's type space Θ_i; Space of type profiles $\Theta := \times_{i \in I} \Theta_i$;

(iv) Player i's utility/payoff function: $u_i : A \times \Theta \to \mathbb{R}$; $u := (u_i)_{i \in I}$; and

(v) Probability distribution over players' type profiles: $\rho \in \Delta(\Theta)$.

All elements are common knowledge, but each player i only knows their own type θ_i, and not the other players' types.

Players privately learn their own type. (WLOG)

Representing Incomplete Information

Definition

- Players have **private values** iff $u_i(a, \theta_i, \theta_{-i}) = u_i(a, \theta_i, \theta'_{-i}) \ \forall \theta_{-i}, \theta'_{-i} \in \Theta_{-i}$. Otherwise, they have **interdependent values**.

Could also consider alternative notions of incomplete information: e.g., uncertainty over what is the strategy set of the opponent.

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- Players have **independent types** iff types are independent across players. Otherwise, they have **correlated types**.

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Representing Incomplete Information

Definition

A pure **strategy** of player i in a Bayesian game is a mapping $s_i : \Theta_i \to A_i$.

Strategy specifies action for each possible type.

Player i's expected payoff: $\tilde{u}_i(s) = \mathbb{E}_{\theta \sim \rho}[u_i(s_1(\theta_1), s_2(\theta_2), ..., s_I(\theta_I), \theta_i, \theta_{-i})]$.

Extend $\tilde{u}_i$ to mixed strategies, $\sigma_i \in \Sigma_i := \Delta(S_i)$.

Representing Incomplete Information

Two classmates, A and B, considering whether to work together.

They work together iff both agree to do so.

If they work alone, payoffs normalised to 0.

If they work together B always gets 10 (improves their grade by 10). However, how much A benefits from working with B depends on B's type.

If B is collaborative (wp α), A also gets a payoff of 10. But if B is a shirker (wp $1 - \alpha$), then A gets a payoff of -6.

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Table: $\theta_B = C$

A \ B	W	N
W	10 , 10	0 , 0
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Table: $\theta_B = S$

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Universal Type Space

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Yes, with a **universal type space** (Mertens & Zamir (1985); Bradenburger & Dekel (1993))

Reassuring that Bayesian games are good tool.

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3. Bayesian Nash Equilibrium
   - Ex-ante vs Interim perspective
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Bayesian Nash Equilibrium

Definition

A **Bayesian Nash Equilibrium** of a Bayesian game $\langle I, A, u, \Theta, \rho \rangle$ is a strategy profile $s = (s_i)_{i \in I}$ such that $\forall i, \forall s_i' \in S_i, \tilde{u}_i(s_i, s_{-i}) \geq \tilde{u}_i(s_i', s_{-i})$.

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NB: consider Bayesian game $\Gamma = \langle I, A, u, \Theta, \rho \rangle$ as standard normal-form game $\tilde{\Gamma} = \langle I, S, \tilde{u} \rangle$.

Set of BNE of Γ is the same as set of NE of $\tilde{\Gamma}$.

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Corollary

For any Bayesian game Γ s.t. $|I|, |A|, |\Theta| < \infty$, $\exists$ Bayesian Nash equilibrium (possibly in mixed strategies).

Ex-ante vs Interim perspective

Ex-ante Perspective:

1. players choose strategies, (distrib. over) mappings from types to actions, to maximise ex-ante expected payoff;
2. types are drawn according to ρ;
3. players learn their own types and play according to their actions;
4. outcomes and payoffs realise.

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Interim perspective:

1. types are drawn according to ρ;
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Arguably more sensible description of a game of incomplete information.

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An ex-interim **Bayesian game** is a tuple $\langle I, A, u, \mathbf{\Theta}, q \rangle$, where

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(iv) Player i's utility/payoff function: $u_i : A \times \mathbf{\Theta} \to \mathbb{R}$; $u := (u_i)_{i \in I}$; and

(v) Ex-interim Belief/Prob. distrib. over opponents' type profiles: $q_i : \mathbf{\Theta}_i \to \Delta(\mathbf{\Theta}_{-i})$.

Ex-Ante vs Interim perspective

Proposition

A strategy profile $\boldsymbol{\sigma}$ is a BNE if and only if $\forall i \in I$ and $\forall \theta_i \in \Theta_i : \rho(\theta_i) > 0$,

$$\mathbb{E}_{\boldsymbol{\theta}_{-i}}[u_i(\sigma_i(\theta_i), \boldsymbol{\sigma}_{-i}(\boldsymbol{\theta}_{-i}), \theta_i, \boldsymbol{\theta}_{-i}) \mid \theta_i] \geq \mathbb{E}_{\boldsymbol{\theta}_{-i}}[u_i(\sigma_i'(\theta_i), \boldsymbol{\sigma}_{-i}(\boldsymbol{\theta}_{-i}), \theta_i, \boldsymbol{\theta}_{-i}) \mid \theta_i], \ \forall \sigma_i' \in \Delta(A_i)^{\Theta_i}.$$

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NB: Possible to find $\rho_i \in \Delta(\Theta) : \rho_i(\theta_{-i} \mid \theta_i) = q_i(\theta_{-i} \mid \theta_i) \quad \forall \theta_i, \theta_{-i}$
(going beyond finite case introduces technical complications).

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Common prior necessary for equivalence between ex-ante BNE and interim BNE.

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A strategy profile σ is an **ex-post Bayesian Nash equilibrium** iff $\forall i$, $u_i(\sigma_i(\theta_i), \sigma_{-i}(\theta_{-i}), \theta) \geq u_i(a_i, \sigma_{-i}(\theta_{-i}), \theta)$, $\forall a_i, \theta$.

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Interpretation: Even if players learn others' types, they would not like to change their actions, given that others are following their strategies.

Note: ex-post BNE yields NE for each game indexed by θ.

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Strategy-Proofness

Closely related to "**Very weak dominance**": $s_i : u_i(s_i(\theta_i), a_{-i}, \theta_i, \theta_{-i}) \geq \tilde{u}_i(a_i, s_{-i}, \theta_i, \theta_{-i})$, $\forall a_i, a_{-i}, \forall \theta$.

- Allows for indifferences.
- Also said **Strategy-proofness**, esp. when $A_i = \Theta_i$.
- You'll hear this term a lot.

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1. Motivation

2. Bayesian Games

3. Bayesian Nash Equilibrium

4. Auctions
   - 2nd-Price Auction
   - Envelope Theorem
   - 1st-Price Auction
   - Revenue Equivalence

5. Purification Theorem

6. Higher-Order Beliefs

7. More

2nd-Price Auction

2nd-Price Auction: winner pays second highest bid.

$u_i(a_i, a_{-i}, v_i) = \mathbf{1}\{i \in \arg\max_j a_j\}(v_i - \max_{j \neq i} a_j)/|\arg\max_j a_j|$

When F_i is degenerate for every i, $a_i = v_i$ is weakly dominant for all players (hence a NE?).

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Independent private values. (What does this mean?)

$v_i \sim F_i$, v_i independent from other types.

$s_i : s_i(v_i) = v_i$ still weakly dominant for all players? Is it a BNE? What does it depend on?

Bayesian Games

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Informationally robust (although perhaps counterintuitive to people.)

Alternative: Ascending auction. People understand it better and play weakly dominant strategy more often.

With good reasons: Obviously Strategy-Proof (Li, 2017 AER)

Roughly, worst case scenario better than best-case scenario from deviation.

Envelope Theorem

What is the envelope theorem?

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- Useful tool to characterise how maximisers change with parameters too!

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Choice set X.

Parameter $t \in [0, 1]$ (think directional derivative in normed vector space)

Objective function: $f : X \times [0, 1] \to \mathbb{R}$.

Value function: $V(t) := \sup_{x \in X} f(x, t)$; Maximisers $X^*(t) := \{x \in X : f(x, t) = V(t)\}$.

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Theorem 1 (Milgrom & Segal 2002 Ecta)

Take any $x^* \in X^*(t)$ and $t \in [0, 1]$, and suppose $f_t'(x^*, t)$ exists.

(1) For $t > 0$, if V is left-differentiable at t, $V'(t^-) \leq f_t'(x^*, t)$.

(2) For $t < 1$, if V is right-differentiable at t, $V'(t^+) \geq f_t'(x^*, t)$.

(3) For $t \in (0, 1)$, if V is differentiable at t, then $V'(t) = f_t'(x^*, t)$.

Envelope Theorem

It would be sufficient to ensure V is differentiable a.e. to get

Theorem 2 (Milgrom & Segal 2002 Ecta)

Take any $x^* \in X^*(t)$ and $t \in [0,1]$, and suppose $f_t'(x^*, t)$ exists.

(1) If $f(x, \cdot)$ is absolutely continuous for all $x \in X$ and there is an integrable function $b : [0,1] \to \mathbb{R}_+$ such that $|f_t'(x,t)| \le b(t)\ \forall x \in X$ and almost all $t \in [0,1]$, then V is absolutely continuous.

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(2) If, in addition, $f(x, \cdot)$ is differentiable for all $x \in X$ and X^* is nonempty-valued a.e. on $[0, 1]$, then for any selection $x^*(t) \in X^*(t)$,

$$V(t) = V(0) + \int_0^t f_t'(x^*(s), s)\, ds$$

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Note: V need not be differentiable everywhere (may have kinks).

1st-Price Auction

Back to auctions: **1st-Price Auction**: winner pays highest bid.

- I bidders with valuations $0 \leq v_i$ and $v_i \sim F$ iid, F atomless and absolutely continuous, bounded support $V_i = [\underline{v}, \overline{v}]$
- **Bids:** $a_i \geq 0$.

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  - Get zero if do not bid highest.
  - Get item if bid highest and pay own bid; uniform tie-breaking.
- NB: $s_i(v_i) = v_i$ is weakly domina**ted** in 1PA!

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Solving for a symmetric PS-BNE $(s^*)_{j \in I}$

Assume: s^* is strictly increasing, differentiable.

s^* strictly increasing + F atomless $\implies$ zero prob. of two identical bids.

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Expected payoff from bidding a_i given type v_i and opponents bidding according to s^*:

$$u(a_i, v_i) = \mathbb{P}(a_i > \max_{j \neq i} s^*(v_j))(v_i - a_i)$$

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Solving for a symmetric PS-BNE $(s^*)_{j \in I}$

Assume: s^* is strictly increasing, differentiable.

s^* strictly increasing + F atomless $\implies$ zero prob. of two identical bids.

Expected payoff from bidding a_i given type v_i and opponents bidding according to s^*:

$$
\begin{aligned}
u(a_i, v_i) &= \mathbb{P}(a_i > \max_{j \neq i} s^*(v_j))(v_i - a_i) \\
&= \mathbb{P}(a_i > s^*(v_j))^{|I|-1}(v_i - a_i) \\
&= F((s^*)^{-1}(a_i))^{|I|-1}(v_i - a_i).
\end{aligned}
$$

1st-Price Auction

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s^* BR to s^*:

$$u(a_i, v_i) = F((s^*)^{-1}(a_i))^{|I|-1}(v_i - a_i).$$

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$$u(a_i, v_i) = F((s^*)^{-1}(a_i))^{|I|-1}(v_i - a_i).$$

$$\implies \quad U(v_i) := u(s^*(v_i), v_i) = F((s^*)^{-1}(s^*(v_i)))^{|I|-1}(v_i - s^*(v_i))$$

$$U(v_i) = F(v_i)^{|I|-1}(v_i - s^*(v_i)). \tag{1}$$

1st-Price Auction

Solving for a symmetric PS-BNE $(s^*)_{j \in I}$

Assume: s^* is strictly increasing, differentiable. (check later)

s^* strictly increasing $\implies$ $s^*(\underline{v})$ wins auction wp0 $\implies$ $U(\underline{v}) = \mathbf{0}$.

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Assume: s^* is strictly increasing, differentiable. (check later)

s^* strictly increasing $\implies$ $s^*(\underline{v})$ wins auction wp0 $\implies$ $U(\underline{v}) = \mathbf{0}$.

(a) $u(a_i, v_i) = F((s^*)^{-\mathbf{1}}(a_i))^{|I|-\mathbf{1}}(v_i - a_i)$ differentiable in v_i.

(b) $U(v_i) = F(v_i)^{|I|-\mathbf{1}}(v_i - s^*(v_i))$ differentiable.

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(a) $u(a_i, v_i) = F((s^*)^{-1}(a_i))^{|I|-1}(v_i - a_i)$ differentiable in v_i.

(b) $U(v_i) = F(v_i)^{|I|-1}(v_i - s^*(v_i))$ differentiable.

Use envelope theorem:

$$U'(v_i) = u'_{v_i}(a_i, v_i)|_{a_i=s^*(v_i)} = F((s^*)^{-1}(a_i))^{|I|-1}|_{a_i=s^*(v_i)} = F(v_i)^{|I|-1}. \tag{2}$$

1st-Price Auction

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Assume: s^* is strictly increasing, differentiable. (check later)

s^* strictly increasing $\implies$ $s^*(\underline{v})$ wins auction wp0 $\implies$ $U(\underline{v}) = \mathbf{0}$.

(a) $u(a_i, v_i) = F((s^*)^{-1}(a_i))^{|I|-1}(v_i - a_i)$ differentiable in v_i.

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Fundamental theorem of calculus:

$$U(v_i) = U(\underline{v}) + \int_{\underline{v}}^{v_i} U'(v)\,\mathrm{d}v = \int_{\underline{v}}^{v_i} U'(v)\,\mathrm{d}v = . \tag{3}$$

Solving for a symmetric PS-BNE $(s^*)_{j\in I}$

Putting it all together:

$$U(v_i) = F(v_i)^{|I|-1}(v_i - s^*(v_i)). \tag{1}$$

$$U'(v_i) = F(v_i)^{|I|-1}. \tag{2}$$

$$U(v_i) = \int_{\underline{v}}^{v_i} U'(v)\, dv. \tag{3}$$

Solving for a symmetric PS-BNE $(s^*)_{j \in I}$

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$$U(v_i) = \int_{\underline{v}}^{v_i} U'(v)\,\mathrm{d}v. \tag{3}$$

$$\implies \quad \int_{\underline{v}}^{v_i} F(v)^{|I|-1}\,\mathrm{d}v = F(v_i)^{|I|-1}(v_i - s^*(v_i))$$

1st-Price Auction

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$$U(v_i) = \int_{\underline{v}}^{v_i} U'(v)\,\mathrm{d}v. \tag{3}$$

$$\Longrightarrow \quad \int_{\underline{v}}^{v_i} F(v)^{|I|-1}\,\mathrm{d}v = F(v_i)^{|I|-1}(v_i - s^*(v_i))$$

$$\Longleftrightarrow \quad s^*(v_i) = v_i - \int_{\underline{v}}^{v_i} \left(\frac{F(v)}{F(v_i)}\right)^{|I|-1} \mathrm{d}v$$

1st-Price Auction

Solving for a symmetric PS-BNE $(s^*)_{j\in I}$

$$s^*(v_i) = v_i - \int_{\underline{v}}^{v_i} \left(\frac{F(v)}{F(v_i)}\right)^{|I|-1} \mathrm{d}v$$

Properties of s^*

Strictly increasing

$$s^*(v' + e) - s^*(v')$$

1st-Price Auction

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Properties of s^*

Strictly increasing

$$s^*(v' + e) - s^*(v')$$
$$= e - \int_{\underline{v}}^{v'+e} \left(\frac{F(v)}{F(v' + e)} \right)^{|I|-1} \mathrm{d}v + \int_{\underline{v}}^{v'} \left(\frac{F(v)}{F(v')} \right)^{|I|-1} \mathrm{d}v$$

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$$= \int_{v'}^{v'+e} 1\,\mathrm{d}v - \int_{\underline{v}}^{v'+e} \left(\frac{F(v)}{F(v'+e)}\right)^{|I|-1} \mathrm{d}v + \int_{\underline{v}}^{v'} \left(\frac{F(v)}{F(v')}\right)^{|I|-1} \left(\frac{F(v'+e)}{F(v'+e)}\right)^{|I|-1} \mathrm{d}v$$

Solving for a symmetric PS-BNE $(s^*)_{j \in I}$

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$$= \int_{v'}^{v'+e} 1 \, \mathrm{d}v - \int_{v'}^{v'+e} \left(\frac{F(v)}{F(v'+e)} \right)^{|I|-1} \mathrm{d}v + \int_{\underline{v}}^{v'} \left(\frac{F(v)}{F(v')} \right)^{|I|-1} \frac{F(v'+e)^{|I|-1} - F(v')^{|I|-1}}{F(v'+e)^{|I|-1}} \mathrm{d}v$$

1st-Price Auction

Solving for a symmetric PS-BNE $(s^*)_{j\in I}$

$$s^*(v_i) = v_i - \int_{\underline{v}}^{v_i} \left(\frac{F(v)}{F(v_i)}\right)^{|I|-1} dv$$

Properties of s^*

Strictly increasing

$$\begin{aligned}
&s^*(v'+e) - s^*(v') \\
&= e - \int_{\underline{v}}^{v'+e} \left(\frac{F(v)}{F(v'+e)}\right)^{|I|-1} dv + \int_{\underline{v}}^{v'} \left(\frac{F(v)}{F(v')}\right)^{|I|-1} dv \\
&= \int_{v'}^{v'+e} 1\, dv - \int_{\underline{v}}^{v'+e} \left(\frac{F(v)}{F(v'+e)}\right)^{|I|-1} dv + \int_{\underline{v}}^{v'} \left(\frac{F(v)}{F(v')}\right)^{|I|-1} \left(\frac{F(v'+e)}{F(v'+e)}\right)^{|I|-1} dv \\
&= \int_{v'}^{v'+e} 1\, dv - \int_{v'}^{v'+e} \left(\frac{F(v)}{F(v'+e)}\right)^{|I|-1} dv + \int_{\underline{v}}^{v'} \left(\frac{F(v)}{F(v')}\right)^{|I|-1} \frac{F(v'+e)^{|I|-1} - F(v')^{|I|-1}}{F(v'+e)^{|I|-1}} dv \\
&\geq \int_{v'}^{v'+e} \frac{F(v'+e)^{|I|-1} - F(v)^{|I|-1}}{F(v'+e)^{|I|-1}} dv + \int_{\underline{v}}^{v'} \left(\frac{F(v)}{F(v')}\right)^{|I|-1} \frac{F(v'+e)^{|I|-1} - F(v')^{|I|-1}}{F(v'+e)^{|I|-1}} dv > 0.
\end{aligned}$$

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Properties of s^*

- Strictly increasing.
- Differentiable (immediate).

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Properties of s^*

- Strictly increasing.
- Differentiable (immediate).
- Bid less than value $s^*(v_i) < v_i$ for $v_i > \underline{v}$. $\implies U(v_i) \geq 0$.

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WT check s^* is optimal.

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Take any $v_i \in V_i$ and $a_i \geq 0$.

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Claim: Given others play s^*, $s^*(v_i)$ does weakly better than a_i $\forall a_i \notin [s^*(\underline{v}), s^*(\overline{v})]$.

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If $a_i > s^*(\overline{v})$, then $u(a_i, v_i) = v_i - a_i < v_i - s^*(\overline{v})$.

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NB: s^* continuous and strictly increasing $\implies \exists! v_i' : a_i = s^*(v_i') \; \forall a_i \in [s^*(\underline{v}), s^*(\overline{v})]$.

$$U(v_i) - u(a_i, v_i) = U(v_i) - U(v_i') + U(v_i') - u(s^*(v_i'), v_i)$$

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$$= U(v_i) - U(v_i') + u(s^*(v_i'), v_i') - u(s^*(v_i'), v_i) = \int_{v_i'}^{v_i} u'_{v_i}(s^*(v), v)\,\mathrm{d}v + \int_{v_i}^{v_i'} u'_{v_i}(s^*(v_i'), v)\,\mathrm{d}v$$

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$$= U(v_i) - U(v_i') + u(s^*(v_i'), v_i') - u(s^*(v_i'), v_i) = \int_{v_i'}^{v_i} u'_{v_i}(s^*(v), v)\,\mathrm{d}v + \int_{v_i}^{v_i'} u'_{v_i}(s^*(v_i'), v)\,\mathrm{d}v$$

$$= \int_{v_i'}^{v_i} \left(F(v)^{|I|-1} - F(v_i')^{|I|-1}\right) \mathrm{d}v.$$

If $v_i > v_i'$, then $F(v) \geq F(v_i')$ for any $v \in [v_i', v_i] \implies U(v_i) - u(a_i, v_i) \geq 0$.

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If $v_i < v_i'$, then

$$U_i(v_i) - u_i(a_i, v_i) = U_i(v_i) - U_i(v_i') + U_i(v_i') - u_i(s^*(v_i'), v_i)$$

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$$\begin{aligned} U_i(v_i) - u_i(a_i, v_i) &= U_i(v_i) - U_i(v_i') + U_i(v_i') - u_i(s^*(v_i'), v_i) \\ &= \int_{v_i'}^{v_i} \left(F(v)^{|I|-1} - F(v_i')^{|I|-1} \right) \mathrm{d}v \end{aligned}$$

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If $v_i < v_i'$, then

$$\begin{aligned} U_i(v_i) - u_i(a_i, v_i) &= U_i(v_i) - U_i(v_i') + U_i(v_i') - u_i(s^*(v_i'), v_i) \\ &= \int_{v_i'}^{v_i} \left(F(v)^{|I|-1} - F(v_i')^{|I|-1}\right) \mathrm{d}v \\ &= \int_{v_i}^{v_i'} \left(F(v_i')^{|I|-1} - F(v)^{|I|-1}\right) \mathrm{d}v \end{aligned}$$

and $F(v_i') \geq F(v)$ for any $v \in [v_i, v_i'] \implies U(v_i) - u(a_i, v_i) \geq 0$.

Revenue Equivalence

Revenue Equivalence Theorem: Any auction setting such that

(i) bidders' types are their valuation, drawn independently from compact convex set,

(ii) the object is allocated to the bidder with the highest valuation,

(iii) a bidder with the lowest possible valuation ($\underline{v}$) gets 0 in expected payoff in equilibrium

generates the same expected revenue to the auctioneer as the 2PA.

Revenue Equivalence

$V^{k:n}$: k-th highest valuation out of I bidders.

$\implies$ Revenue in 2PA: $V^{2:I}$.

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$V^{k:n}$: k-th highest valuation out of I bidders.

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Bids in 1PA:

$$s^*(v) = v - \int_{\underline{v}}^{v} \left(\frac{F(s)}{F(v)}\right)^{I-1} \mathrm{d}s = \frac{1}{F^{1:I-1}(v)}\left[F^{1:I-1}(v)v - \int_{\underline{v}}^{v} F^{1:I-1}(s)\,\mathrm{d}s\right]$$

$$= \frac{1}{F^{1:I-1}(v)}\int_{\underline{v}}^{v} s\,\mathrm{d}F^{1:I-1}(s) \qquad \text{(Integration by parts)}$$

$$= \mathbb{E}[V^{1:I-1} | V^{1:I-1} < v]$$

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$$= \frac{1}{F^{1:I-1}(v)} \int_{\underline{v}}^{v} s\,\mathrm{d}F^{1:I-1}(s) \qquad \text{(Integration by parts)}$$

$$= \mathbb{E}[V^{1:I-1} | V^{1:I-1} < v]$$

$\implies$ Revenue in 1PA:

$$s^*(V^{1:I}) = \mathbb{E}[V^{1:I-1} | V^{1:I-1} < V^{1:I}]$$

Revenue Equivalence: $\mathbb{E}[V^{2:I}] = \mathbb{E}[\mathbb{E}[V^{1:I-1} | V^{1:I-1} < V^{1:I}]]$

Overview

Purification Theorem

MSNE hard to justify: although player is indifferent, they need to randomise in very particular way to make opponents indifferent as well.

Purification: Harsanyi (1973) provided a justification for MSNE of a normal-form game $\Gamma = \langle I, S, u \rangle$ as a limit case of perturbed games.

Suppose true preference is unobserved by opponents (random) and given by

$$\tilde{u}_i(s, \theta_i) := u_i(s) + \varepsilon \theta_i^S$$

where θ_i^S are independent across players and drawn from a distribution F_i with density f_i.

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where θ_i^S are independent across players and drawn from a distribution F_i with density f_i.

Theorem

Fix a finite set of players I and strategy spaces S_i. For almost all payoff vectors $u = (u_i)_{i \in I}$ and for all independent and twice-differentiable densities f_i on $[-1, 1]^{|S|}$, any mixed strategy Nash equilibrium of the normal-form game $\Gamma = \langle I, S, u \rangle$ is the limite of a sequence of pure strategy Bayesian Nash equilibria of the Bayesian game with perturbed payoffs $(\tilde{u}_i)_{i \in I}$.

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Suppose true preference is unobserved by opponents (random) and given by

$$\tilde{u}_i(s, \theta_i) := u_i(s) + \varepsilon \theta_i^S$$

where θ_i^S are independent across players and drawn from a distribution F_i with density f_i.

Theorem

Fix a finite set of players I and strategy spaces S_i. For almost all payoff vectors $u = (u_i)_{i \in I}$ and for all independent and twice-differentiable densities f_i on $[-1, 1]^{|S|}$, any mixed strategy Nash equilibrium of the normal-form game $\Gamma = \langle I, S, u \rangle$ is the limite of a sequence of pure strategy Bayesian Nash equilibria of the Bayesian game with perturbed payoffs $(\tilde{u}_i)_{i \in I}$.

Note the limits of the result: "for *almost* all payoff vectors"

Higher-Order Beliefs

Coordination Game (bank runs, currency attacks)

		Col Player	
		Invest	Not Invest
Row Player	Invest	θ,θ	$\theta - 1$,0
	Not Invest	0,$\theta - 1$	0,0

Complete Information. NE?

$\theta < 0$: Not invest is strictly dominant and (NI,NI) the unique NE.

$\theta > 1$: Invest is strictly dominant and (I,I) the unique NE.

$\theta \in [0, 1]$: (NI,NI), (I,I), and mixed is NE.

Higher-Order Beliefs

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	Not Invest	$0,\theta - 1$	0,0

Incomplete Information

Suppose both players observe a signal about the state θ.

$\theta_i := \theta + \varepsilon_i$, $\varepsilon_i \sim N(0, \sigma^2)$ iid.

$\theta \mid \theta_i \sim N(\theta_i, \sigma^2)$, because $\theta = \theta_i - \varepsilon_i$

$\theta_j \mid \theta_i := \theta \mid \theta_i + \varepsilon_j \mid \theta_i = \theta \mid \theta_i + \varepsilon_j \sim N(\theta_i, 2\sigma^2)$.

Higher-Order Beliefs

Coordination Game (bank runs, currency attacks)

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$\theta_j \mid \theta_i := \theta \mid \theta_i + \varepsilon_j \mid \theta_i = \theta \mid \theta_i + \varepsilon_j \sim N(\theta_i, 2\sigma^2)$.

Implicitly, this is saying that players have uninformative or improper prior on θ that is uniform over the real line.

Why improper? because there is no uniform distribution over the real line; it cannot add-up to one if it has a constant pdf.

Higher-Order Beliefs

Incomplete Information

$\theta \mid \theta_i \sim N(\theta_i, \sigma^2)$ and $\theta_j \mid \theta_i \sim N(\theta_i, 2\sigma^2)$.

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$\theta \mid \theta_i \sim N(\theta_i, \sigma^2)$ and $\theta_j \mid \theta_i \sim N(\theta_i, 2\sigma^2)$.

Claim: $s_i(\theta_i) := 1\{\theta_i > 1/2\}$ is an equilibrium.

- Given $s_j = 1\{\theta_j > 1/2\}$, player i's payoff to investing conditional on θ_i and s_j is

$$\theta_i - \mathbb{P}\left(\theta_j \leq 1/2 \mid \theta_i\right) = \theta_i - \Phi\left(\frac{1/2 - \theta_i}{\sqrt{2}\sigma}\right)$$

  strictly increasing in θ_i and zero when $\theta_i = 1/2$.
- $s_i(\theta_i) := 1\{\theta_i > 1/2\}$ is the unique best response to s_j.

Higher-Order Beliefs

Incomplete Information

$\theta \mid \theta_i \sim N(\theta_i, \sigma^2)$ and $\theta_j \mid \theta_i \sim N(\theta_i, 2\sigma^2)$.

WTS **Proposition**: In any eqm, $s_i(\theta_i) = \mathbf{1}$ a.e. on $(1/2, \infty)$ and $s_i(\theta_i) = \mathbf{0}$ a.e. on $(-\infty, 1/2)$.

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Preliminaries: Define $f(\theta_i, \tilde{\theta}) := \theta_i - \Phi\left(\frac{\tilde{\theta}-\theta_i}{\sqrt{2}\sigma}\right)$.

Note: $\mathbb{P}(\theta_j < \tilde{\theta} \mid \theta_i) = \mathbb{P}\left(\frac{\theta_j-\theta_i}{\sqrt{2}\sigma} < \frac{\tilde{\theta}-\theta_i}{\sqrt{2}\sigma} \mid \theta_i\right) = \Phi\left(\frac{\tilde{\theta}-\theta_i}{\sqrt{2}\sigma}\right)$.

$f(\theta_i, \tilde{\theta})$ is continuous in $(\theta_i, \tilde{\theta})$, strictly increasing in θ_i, and strictly decreasing in $\tilde{\theta}$.

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Claim: $\forall\theta_i > 1$, $\mathbb{E}[u_i(I, s_j(\theta_j), \theta)|\theta_i] = \theta_i - \mathbb{E}[s_j(\theta_j)|\theta_i] > 0$.

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- Note that: $\forall \delta > 0$, $f(\theta_i, \tilde{\theta}) > \delta \; \forall \theta_i \geq 1 + \delta$ and $\forall \tilde{\theta}$.
- Then $\forall \theta_i > 1$, $\mathbb{E}[u_i(I, s_j(\theta_j), \theta)|\theta_i] = \theta_i - \mathbb{E}[s_j(\theta_j)|\theta_i] \geq \theta_i = f(\theta_i, -\infty) > 0$ (where $f(\theta_i, -\infty) := \lim_{\tilde{\theta} \to -\infty} f(\theta_i, \tilde{\theta})$).

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For $k = 1, 2, ...$, define $\overline{\theta}^{k+1} := \inf\{\theta_i | f(\theta_i, \overline{\theta}^k) > 0\}$, where $\overline{\theta}^1 = 1$.

Incomplete Information

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For $k = 1, 2, ...$, define $\overline{\theta}^{k+1} := \inf\{\theta_i | f(\theta_i, \overline{\theta}^k) > 0\}$, where $\overline{\theta}^1 = 1$.

Claim: $\overline{\theta}^k > \overline{\theta}^{k+1}$ $\forall k$.

- True for $k = 0$.
- Induction: $\overline{\theta}^{k+1} < \overline{\theta}^k \implies 0 = f(\overline{\theta}^{k+2}, \overline{\theta}^{k+1}) = f(\overline{\theta}^{k+1}, \overline{\theta}^k) < f(\overline{\theta}^{k+1}, \overline{\theta}^{k+1})$
  $\because f$ strictly decreasing in 2nd argument and $\overline{\theta}^{k+1} < \overline{\theta}^k$.
- $f(\overline{\theta}^{k+2}, \overline{\theta}^{k+1}) < f(\overline{\theta}^{k+1}, \overline{\theta}^{k+1}) \implies \overline{\theta}^{k+2} < \overline{\theta}^{k+1}$ $\because f$ strictly increasing in 1st argument.

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Claim: (Induction step) If $s_j(\theta_j) = 1 \forall \theta_j > \overline{\theta}^k$, then $\forall \theta_i > \overline{\theta}^{k+1}$ $\mathbb{E}[u_i(I, s_j(\theta_j), \theta)|\theta_i] > 0$.

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For $k = 1, 2, \dots$, define $\overline{\theta}^{k+1} := \inf\{\theta_i | f(\theta_i, \overline{\theta}^k) > 0\}$, where $\overline{\theta}^1 = 1$.

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- $\forall \theta_i > \overline{\theta}^{k+1}$, $\mathbb{E}[u_i(I, s_j(\theta_j), \theta)|\theta_i] \geq f(\theta_i, \overline{\theta}^k) > 0 = f(\overline{\theta}^{k+1}, \overline{\theta}^k)$
  $\because$ f is strictly increasing in 1st arg.

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Claim: At any BNE s, $s_i(\theta_i) = 1$ a.e. on $\theta_i > \overline{\theta}^k$, for all k and $i = 1, 2$.

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Claim: At any BNE s, $s_i(\theta_i) = 1$ a.e. on $\theta_i > \overline{\theta}^k$, for all k and $i = 1, 2$.

- True for $k = 1$, $\mathbb{E}[u_i(I, s_j(\theta_j), \theta)|\theta_i] \geq \theta_i - 1 > 0$ for any $\theta_i > \overline{\theta}^1 = 1$, no matter s_j.
- Then, for any $s_i' : s_i'(\theta_i) \neq 1$ for a positive measure of $\theta_i > \overline{\theta}^1$ is strictly dominated by s_i s.t. $s_i = s_i'$ on $(-\infty, \overline{\theta}^1]$ and $s_i = 1$ on $(\overline{\theta}^1, \infty)$.
- Iterating the argument, for any k, for any $s_i' : s_i'(\theta_i) \neq 1$ for a positive measure of $\theta_i > \overline{\theta}^k$ is iteratedly strictly dominated by s_i s.t. $s_i = s_i'$ on $(-\infty, \overline{\theta}^k]$ and $s_i = 1$ on $(\overline{\theta}^k, \infty)$.

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Claim: $\forall \theta_i > 1$, $\mathbb{E}[u_i(I, s_j(\theta_j), \theta)|\theta_i] = \theta_i - \mathbb{E}[s_j(\theta_j)|\theta_i] > 0$.

For $k = 1, 2, ...$, define $\overline{\theta}^{k+1} := \inf\{\theta_i | f(\theta_i, \overline{\theta}^k) > 0\}$, where $\overline{\theta}^1 = 1$.

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Claim: At any BNE s, $s_i(\theta_i) = 1$ a.e. on $\theta_i > \overline{\theta}^k$, for all k and $i = 1, 2$.

Claim: $\overline{\theta}^k > 0$ $\forall k$.

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Incomplete Information

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Claim: At any BNE s, $s_i(\theta_i) = 1$ a.e. on $\theta_i > \overline{\theta}^k$, for all k and $i = 1, 2$.

Claim: $\overline{\theta}^k > 0\ \forall k$.

- Note that: $\forall \delta > 0$, $f(\theta_i, \tilde{\theta}) < -\delta\ \forall \theta_i \leq -\delta$ and $\forall \tilde{\theta}$.
- Then $\forall \theta_i < 0$, $\mathbb{E}[u_i(I, s_j(\theta_j), \theta)|\theta_i] = \theta_i - \mathbb{E}[s_j(\theta_j)|\theta_i] \leq \theta_i = f(\theta_i, \infty) < 0$ (where $f(\theta_i, \infty) := \lim_{\tilde{\theta}\to\infty} f(\theta_i, \tilde{\theta})$).
- Then, as $f(\overline{\theta}^k, \overline{\theta}^k) > 0 > f(0, \overline{\theta}^k)$ and f is strictly increasing in 1st argument, then $\overline{\theta}^k > 0\ \forall k$.

Higher-Order Beliefs

Incomplete Information

$\theta \mid \theta_i \sim N(\theta_i, \sigma^2)$ and $\theta_j \mid \theta_i \sim N(\theta_i, 2\sigma^2)$.

WTS **Proposition**: In any eqm, $s_i(\theta_i) = 1$ a.e. on $(1/2, \infty)$ and $s_i(\theta_i) = 0$ a.e. on $(-\infty, 1/2)$.

Preliminaries: Define $f(\theta_i, \tilde{\theta}) := \theta_i - \Phi\left(\frac{\tilde{\theta}-\theta_i}{\sqrt{2}\sigma}\right)$.

Claim: $\forall \theta_i > 1, \mathbb{E}[u_i(I, s_j(\theta_j), \theta)|\theta_i] = \theta_i - \mathbb{E}[s_j(\theta_j)|\theta_i] > 0$.

For $k = 1, 2, \ldots$, define $\overline{\theta}^{k+1} := \inf\{\theta_i | f(\theta_i, \overline{\theta}^k) > 0\}$, where $\overline{\theta}^1 = 1$.

Claim: $\overline{\theta}^k > \overline{\theta}^{k+1} \, \forall k$.

Claim: (Induction step) If $s_j(\theta_j) = 1 \forall \theta_j > \overline{\theta}^k$, then $\forall \theta_i > \overline{\theta}^{k+1}$ $\mathbb{E}[u_i(I, s_j(\theta_j), \theta)|\theta_i] > 0$.

Claim: At any BNE s, $s_i(\theta_i) = 1$ a.e. on $\theta_i > \overline{\theta}^k$, for all k and $i = 1, 2$.

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- $\{\overline{\theta}^k\}_k$ decreasing sequence, bounded below by 0 $\implies$ it converges to some $\overline{\theta}^\infty \geq 0$, by monotone convergence theorem.
- $0 = \lim_{k\to\infty} f(\overline{\theta}^{k+1}, \overline{\theta}^k) = f(\overline{\theta}^\infty, \overline{\theta}^\infty) = \overline{\theta}^\infty - \Phi\left(\frac{\overline{\theta}^\infty - \overline{\theta}^\infty}{\sqrt{2}\sigma}\right) = \overline{\theta}^\infty - 1/2$.

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Fully symmetric arguments:

Claim: $\forall \theta_i < 0$, $\mathbb{E}[u_i(I, s_j(\theta_j), \theta)|\theta_i] = \theta_i - \mathbb{E}[s_j(\theta_j)|\theta_i] < 0$.

For $k = 1, 2, ...$, define $\underline{\theta}^{k+1} := \sup\{\theta_i | f(\theta_i, \underline{\theta}^k) < 0\}$, where $\underline{\theta}^1 = 0$.

Claim: $\underline{\theta}^k < \underline{\theta}^{k+1}\ \forall k$.

Claim: (Induction step) If $s_j(\theta_j) = 0 \forall \theta_j < \underline{\theta}^k$, then $\forall \theta_i < \underline{\theta}^{k+1}$ $\mathbb{E}[u_i(I, s_j(\theta_j), \theta)|\theta_i] < 0$.

Claim: At any BNE s, $s_i(\theta_i) = 0$ a.e. on $\theta_i < \underline{\theta}^k$, for all k and $i = 1, 2$.

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Proposition: In any eqm, $s_i(\theta_i) = 1$ a.e. on $(1/2, \infty)$ and $s_i(\theta_i) = 0$ a.e. on $(-\infty, 1/2)$.

NB: proposition holds $\forall \sigma$. Taking $\sigma \downarrow 0$ selects unique NE.

Global game approach to selection of NE.

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Iterating the argument on players' beliefs has higher-order beliefs working in the background to refine what the opponent will do.

Common knowledge of rationality is doing all the heavy-lifting in determining how players behave!

Overview

1. Motivation
2. Bayesian Games
3. Bayesian Nash Equilibrium
4. Auctions
5. Purification Theorem
6. Higher-Order Beliefs
7. More

More

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Experiments: Winner's curse Charness & Levin (2009 AEJMicro), Overbidding and QRE: Goeree, Holt, & Palfrey (2002 JET); Camerer, Nunnari, & Palfrey (2016 GEB); and Charness, Levin, & Schmeidler (2019 JET).