3. Optimal Choice and Consumer Theory

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MRes Microconomics

Overview

1. Consumption
2. Utility Maximisation Problem
3. Expenditure Minimisation Problem
4. Solving Optimisation Problems using Calculus
5. Afriat's Theorem
6. More

Overview

Consumer's Problem

Modelling demand: one of the first problems of economics

- Cournot, Walras, Menger, Jevons, Pareto, Marshall, Samuelson, Hicks, Debreu, Arrow, Stiegler, etc.

Today: classical consumer theory

- a straightforward application of what we've seen

Overview

Utility Maximisation Problem

Bundles of goods: $X = \mathbb{R}^k_+$

Preference relation: $\succsim \subseteq X^2$, Utility function: $u : X \to \mathbb{R}$ represents $\succsim$ (assumed $\exists$)

Prices: $p \in \mathbb{R}_{++}$, Income: $w \geq 0$ Budget constraint: $B(p, w) := \{x \in X \mid p \cdot x \leq w\}$

Definition (Utility Maximisation Problem)

$$x(p, w) := \arg\max_{\succsim} B(p, w) = \arg\max_{x \in B(p,w)} u(x), \qquad v(p, w) := \sup_{x \in B(p,w)} u(x) \qquad \text{(UMP)}$$

(Marshallian) Demand: $x(p, w) \subseteq B(p, w)$; set of maximisers

Indirect Utility: $v(p, w)$; maximised utility

General Properties

Proposition

$v(p, w)$ is quasiconvex in (p, w), weakly decreasing in p, and weakly increasing in w.

Proof

(1) WTS quasiconvexity.

Take any $(p, w), (p', w') \in \{(p, w) \mid v(p, w) \leq \bar{v}\}$ and $\lambda \in [0, 1]$.
Let $(p'', w'') := \lambda(p, w) + (1 - \lambda)(p', w')$.

WTS $v(p'', w'') \leq \max\{v(p, w), v(p', w')\}, \forall \lambda \in [0, 1]$.

- WTS $\forall x'' \in X : p'' \cdot x'' \leq w''$, (i) $x'' \in B(p, w)$ or (ii) $x'' \in B(p', w')$.
  Suppose not: Then $p \cdot x'' > w$ and $p' \cdot x'' > w'$
  $\implies p'' \cdot x'' = (\lambda p + (1 - \lambda)p') \cdot x'' > \lambda w + (1 - \lambda)w' = w''$
  $\implies x'' \notin B(\lambda(p, w) + (1 - \lambda)(p', w'))$, contradiction.
- Hence, $x'' \in B(p, w) \implies u(x'') \leq v(p, w) \leq \max\{v(p, w), v(p', w')\}$
  or $x'' \in B(p', w') \implies u(x'') \leq v(p', w') \leq \max\{v(p, w), v(p', w')\}$.

(2) WTS v is weakly decreasing in p, and weakly increasing in w.

$p \geq p', w \leq w' \implies B(p, w) \subseteq B(p', w') \implies v(p, w) \leq v(p', w')$ (why?). $\square$

General Properties

Proposition

$v(p, w)$ and $x(p, w)$ are homogeneous of degree zero in (p, w): $\forall \lambda > \mathbf{0}$, $v(\lambda p, \lambda w) = v(p, w)$ and $x(\lambda p, \lambda w) = x(p, w)$.

Proof

As $B(\lambda p, \lambda w) = B(p, w)$, then $\arg\max_{\succsim} B(p, w) = \arg\max_{\succsim} B(\lambda p, \lambda w)$. $\square$

If you scale up prices and income, then the consumer is able to afford exactly the same bundles. Both indirect utility and maximisers remain the same.

Money neutrality!

Implications of Continuity

Proposition

If $\succsim$ is continuous, then $x(p, w)$ is nonempty.

Correspondences: A Refresher

Definition

A **correspondence** F from X to Y is a mapping that associates with each element $x \in X$ a subset $A \subseteq Y$, denoted by $F : X \rightrightarrows Y$ or $F : X \to \mathbf{2}^Y$, with $F(x) \subseteq Y$.
For $A \subseteq X$, define the image of F as $F(A) := \cup_{x \in A} F(x)$.

Definition

Let (X, d_X) and (Y, d_Y) be metric spaces and $F : X \rightrightarrows Y$. F is

(i) **upper hemicontinuous (uhc) at** $x_0 \in X$ iff $\forall$ open set $U \subseteq Y$, s.t. $F(x_0) \subseteq U$, $\exists \varepsilon > 0 : F(B_\varepsilon(x_0)) \subseteq U$;

(ii) **upper hemicontinuous (uhc)** if it is upper hemicontinuous at any $x_0 \in X$;

(iii) **lower hemicontinuous (lhc) at** $x_0 \in X$ if $\forall$ open set $U \subseteq Y$, s.t. $F(x_0) \cap U \neq \emptyset$, $\exists \varepsilon > 0 : F(x) \cap U \neq \emptyset, \forall x \in B_\varepsilon(x_0)$;

(iv) **lower hemicontinuous (lhc)** iff it is lower hemicontinuous at any $x_0 \in X$;

(v) **continuous at** $x_0 \in X$ if it is both uhc and lhc at x_0;

(vi) **continuous** if it is both uhc and lhc.

Correspondences: A Refresher

Proposition

Let (X, d_X) and (Y, d_Y) be metric spaces and $F : X \rightrightarrows Y$. F is

(i) lhc at x_0 if and only if $\forall$ sequence $\{x_n\}_n \subseteq X : x_n \to x_0$ and $\forall y_0 \in F(x_0)$, there is N and a sequence $\{y_n\}_{n>N}$ sat. $y_n \in F(x_n)$, s.t. $y_n \to y_0$.

(ii) uhc (and *compact-valued*) at x_0 if (and only if) $\forall$ sequence $\{x_n\}_n \subseteq X : x_n \to x_0$ and $\forall$ sequence $\{y_n\}_n : y_n \in F(x_n)$, $\exists$ subsequence $\{y_m\}_m \subseteq \{y_n\}_n$ s.t. $y_m \to y_0 \in F(x_0)$.

Part (i) says lhc = every point $y_0 \in F(x_0)$ can be reached by some sequence $y_n \in F(x_n)$.

Part (ii) that uhc and compact-valuedness = limit y_0 of converging sequences $y_n \in F(x_n)$ is point in limitting set $F(x_0)$.

Read lecture notes on correspondences.

Implications of Continuity (Cont'd)

Berge's Maximum Theorem

Let X and Θ be metric spaces, $f : X \times \Theta \to \mathbb{R}$ be a continuous function, and $B : \Theta \rightrightarrows X$ be a non-empty and compact-valued correspondence.
Let $f^*(\theta) := \sup_{x \in B(\theta)} f(x, \theta)$ and $X^*(\theta) := \arg\sup_{x \in B(\theta)} f(x, \theta)$.

If B is continuous at $\theta \in \Theta$, then f^* is continuous at θ and X^* is uhc, nonempty, and compact-valued at θ.

Very powerful stuff that can be applied off-the-shelf!

Proposition

If $\succsim$ is continuous, then $x(p, w)$ is upper hemicontinuous, nonempty- and compact-valued in (p, w).
Further, if u is a continuous u-representation of $\succsim$, $v(p, w)$ is continuous.

(Proof left as an exercise.)

Proposition

If $\succsim$ is convex, then $x(p, w)$ is convex. If $\succsim$ is strictly convex, then $x(p, w)$ contains at most one element.

Corollary

If $\succsim$ is continuous and strictly convex, then $x(p, w)$ is continuous in (p, w)/

Implications of Local Non-Satiation

Proposition (Walras's Law)

If $\succsim$ is locally non-satiated, then for any $x \in x(p,w)$, and any $(p,w) \in \mathbb{R}^k_{++} \times \mathbb{R}_+$, $p \cdot x = w$.

Proof

Let $x \in x(p,w)$; suppose $p \cdot x < w$. $\exists \varepsilon > \mathbf{0} : \forall x' \in B_\varepsilon(x), p \cdot x' < w$.

Local nonsatiation $\implies$ $\exists x'' \in B_\varepsilon(x) : x'' \succ x$.

As $x'' \in B(p,w)$, then $x \notin \arg\max_{\succsim} B(p,w)$. $\square$

Implications of Local Non-Satiation

Proposition

If u is continuous and locally nonsatiated, then $v(p, w)$ is strictly increasing in w.

Proof

$w < w' \implies B(p, w) \subsetneq B(p, w')$. Take $x \in x(p, w)$ and $x' \in x(p, w')$ (which exist; why?).

$x \in x(p, w) \subseteq B(p, w) \implies p \cdot x \leq w < w'$, and therefore it violates Walras's Law.

Hence, $x \notin \arg\max_{\succsim} B(p, w') \ni x' \implies x' \succ x \iff v(p, w') = u(x') > u(x) = v(p, w)$. $\square$

Implications of Homotheticity

Proposition

Let every consumer $i \in I$ have income $w_i \geq 0$ and identical preferences $\succsim$. If $\succsim$ is continuous, homothetic and strictly convex, then $\sum_{i \in I} x(p, w_i) = x(p, \sum_{i \in I} w_i)$.

Simple aggregation result!

Proof

$\succsim$ homothetic $\implies x \in x(p, 1) \iff w \cdot x \in x(p, w)$.

$\succsim$ strictly convex $\implies |x(p, w)| \leq 1$.

$\succsim$ continuous $\implies x(p, w) \neq \emptyset$.

$\implies \sum_{i \in I} x(p, w_i) = \sum_{i \in I} w_i \cdot x(p, 1) = x(p, \sum_{i \in I} w_i)$.

$\square$

Overview

Expenditure Minimisation Problem

'Dual problem' of UMP: given a utility level u, minimise expenditure, subject to attaining at least a prespecified utility threshold

$U := \text{co}(u(X))$
(convex hull of A: smallest convex set that contains A)

Fix $u \in U \subseteq \mathbb{R}$

Definition (Expenditure Minimisation Problem)

$$h(p,u) := \arg\min_{x \in X \,|\, u(x) \geq u} p \cdot x, \qquad e(p,u) := \inf_{x \in X \,|\, u(x) \geq u} p \cdot x \qquad \text{(EMP)}$$

(Hicksian) Demand: $h(p,u) \subseteq X$; set of minimisers

Expenditure Function: $e(p,u)$

General Implications

Proposition

h is homogeneous of degree zero in p.
e is homogeneous of degree one in p.

By definition: $\forall \lambda > 0$, $h(\lambda p, u) = h(p, u)$ and $e(\lambda p, u) = \lambda e(p, u)$. $\square$

General Implications

Proposition

e is concave in p.

(Immediately: e is the infimum over concave functions... But, direct proof:)

Fix $p, p' \in \mathbb{R}^k_{++}$, $u \in U$, and $\lambda \in [0, 1]$. Let $p'' := \lambda p + (1 - \lambda)p'$ and $A := \{x \in X \mid u(x) \geq u\}$.

$\forall x \in A$, (i) $p \cdot x \geq \inf_{x \in A} p \cdot x =: e(p, u)$ and (ii) $p' \cdot x \geq e(p', u)$.

$\implies \forall x \in A, \; (\lambda p + (1 - \lambda)p') \cdot x \geq \lambda e(p, u) + (1 - \lambda)e(p', u)$.

$\implies e(\lambda p + (1 - \lambda)p', u) := \inf_{x \in A}(\lambda p + (1 - \lambda)p') \cdot x \geq \lambda e(p, u) + (1 - \lambda)e(p', u)$.

□

Supergradient

Definition

$c \in \mathbb{R}^k$ is **supergradient** of $f : X \to \mathbb{R}$ at $x_0 \in X$ iff $f(y) \leq f(x_0) + c \cdot (y - x_0), \forall y \in X$.
Set of supergradients/superdifferential of f at x_0 is denoted by $\partial f(x_0)$.

Theorem

Let $X \subseteq \mathbb{R}^k$ be a convex set and f be a real-valued function on X. f is concave on $\text{int}(X)$ if and only if $\forall x \in \text{int}(X), \partial f(x) \neq \emptyset$.

Intuition:

- Pick $x, y, z \in X$. For $c \in \partial f(x)$, $f(y) \leq f(x) + c \cdot (y - x)$ and $f(z) \leq f(x) + c \cdot (z - x)$.
- By convex combination of the two, with $\lambda \in (0, 1)$,
  $\lambda f(y) + (1 - \lambda) f(z) \leq f(x) + c(\lambda y + (1 - \lambda) z - x)$.
- Choosing $x = \lambda y + (1 - \lambda) z$ delivers concavity of f.

Generalises notion of derivative to functions not necessarily differentiable everywhere;
e.g., $f(x) := -|x|$.

Properties of Concave Functions

We can say a lot about concave functions:

Proposition

(i) For any $x \in \text{relint}(X)$, $\partial f(x)$ is nonempty, convex, and compact.

(Relative interior of a convex set A, $\text{relint}(A) := \{x \in A \mid \forall y \in A \setminus \{x\}, \exists z \in A, \lambda \in (\mathbf{0}, \mathbf{1}) \text{ s.t. } x = \lambda y + (\mathbf{1} - \lambda)z\}$.)

(ii) For any $c \in \partial f(x)$ and $c' \in \partial f(x')$, $(c' - c) \cdot (x' - x) \leq \mathbf{0}$.

(iii) If f is continuous at x, then the superdifferential $\partial f(x)$ is a singleton if and only if f is differentiable at x. In this case, $f'(x) = c \in \partial f(x) = \{c\}$.

(iv) f'' exists almost everywhere in $\text{int}(X)$ (Alexandrov theorem).

(v) If $k = \mathbf{1}$, at any $x \in \text{int}X$, $\partial f(x) = [f'_+(x), f'_-(x)]$, where f'_-, f'_+ denote the left- and right-derivatives of f.

Hicksian Demand

Lemma

If $x_0 \in h(p_0, u)$, then x_0 is a supergradient of $e(\cdot, u)$ at p_0.

Proof

As $p_0 \cdot x_0 = e(p_0, u)$ and $p \cdot x_0 \geq e(p, u)$, then, $\forall p \in \mathbb{R}^k_{++}$, we have $e(p, u) \leq e(p_0, u) + x_0 \cdot (p - p_0)$. $\square$

Theorem (Compensated Law of Demand)

If $p' \geq p, x \in h(p, u)$, and $x' \in h(p', u)$, then $(p' - p) \cdot (x' - x) \leq \mathbf{0}$.

Proof

Follows immediately from property (ii) of concave functions and the fact that Hicksian demand is a supergradient of e. $\square$

If $p'_j = p_j \; \forall j \neq i$ and $p'_i > p_i$, then Hicksian demand sat. $x'_i \leq x_i$.

Monotonicity

Proposition

e is weakly increasing in p and u.

Proof

Take $u' \geq u$ and $p' \geq p$.

- $\forall p'' \in \mathbb{R}^k_{++}$ transitivity implies $\{x \in X \mid u(x) \geq u\} \supseteq \{x \in X \mid u(x) \geq u'\} \implies e(p'', u) \leq e(p'', u')$.
- $\forall u'' \in U, p \cdot x \leq p' \cdot x \; \forall x : u(x) \geq u''$, which implies $e(p, u'') \leq e(p', u'')$. $\square$

Implications of Continuity

Proposition

If u is continuous, then $e(p, u)$ is continuous and $h(p, u)$ is nonempty, compact-valued, and uhc in (p, u).

(Proof left as an exercise.)

Lemma

If u is continuous, then $\forall x \in h(p, u)$, $u(x) = u$.

Proof

Suppose $u(x) > u$. Continuity $\implies \exists \lambda \in [0, 1) : u(\lambda x) > u$.
But then $p \cdot x > p \cdot \lambda x$ and $u(\lambda x) > u \implies x \notin h(p, u)$, a contradiction. □

Compensated demand: how the consumer substitutes across the different goods while attaining the same utility level.

Implications of Local Non-Satiation

Theorem

Let $\succsim$ be locally nonsatiated and u be a continuous utility representation of $\succsim$. Then

(i) $h(p, v(p, w)) = x(p, w)$ and $e(p, v(p, w)) = w$;

(ii) $h(p, u) = x(p, e(p, u))$ and $u = v(p, e(p, u))$.

(Proof left as an exercise.)

Connect Marshallian and Hicksian demand!

Compensated demand: Increase in prices; how much money needed to keep utility constant at u? $e(p, u)$

Implications of Convexity

Proposition

(i) If $\succsim$ is convex, then $h(p,u)$ is convex.

(ii) If $\succsim$ is strictly convex and u is continuous, then $h(p,u)$ is a singleton, continuous in (p,u), and $h(p,u) = e'_p(p,u)$.

Proof

(i) Fix $x, x' \in h(p,u)$ and $\lambda \in [0,1]$.

$p \cdot (\lambda x + (1-\lambda)x') = e(p,u)$ and $u(\lambda x + (1-\lambda)x') \geq \min\{u(x), u(x')\} \geq u$

$\implies \lambda x + (1-\lambda)x' \in h(p,u)$.

(ii) From LNS (why?) and continuity, $x(p, e(p,u)) = h(p,u)$. Given, in addition, $\succsim$ convex, then $x(p, e(p,u))$ is singleton.

Continuity follows uhc + singleton, uhc from Berge's Maximum Theorem.

$h(p,u) = e'_p(p,u)$ follows $h(p,u)$ being the unique supergradient of $e(p,u)$.

$\square$

Solving Optimisation Problems using Calculus

You are expected to be able to handle constrained optimisation problems using Lagrangian methods and Karush-Kuhn-Tucker conditions.

Overview

Afriat's Theorem

Consumer Choice in the Wild

Dataset: $\mathcal{D} = \{(x_t, p_t)\}_{t=1,...,T}$

Question: can data be rationalised by utility-maximising consumer behaviour?

i.e., $\exists x(\cdot, \cdot) : \forall t = 1, ..., T, x_t \in x(p_t, w_t)$ for some income w_t?

No income? Assume $\succsim$ sat. LNS $\implies w_t = p_t \cdot x_t$.

Revealed Preference

Adjust GARP to consumer demand problem:

Definition

(i) x is **directly revealed preferred to** x' if x was chosen and x' was affordable under p: $p \cdot x' \leq p \cdot x$.

(ii) x is **revealed preferred** to x' if $\exists \{x_m\}_{m=1,\dots,M}$ s.t. $x = x_1, x' = x_M$ and for $i = 1, \dots, M-1$, x_i is directly revealed preferred to x_{i+1}.

(iii) x is **revealed strictly preferred to** x' if it was strictly less expensive than x under p: $p \cdot x' < p \cdot x$.

Definition

The dataset $\mathcal{D} = \{(x_t, p_t)\}_{t=1,\dots,T}$ satisfies **Generalised Axiom of Revealed Preference** (GARP) iff there are no x, x' s.t. x is revealed preferred to x' and x' is revealed strictly preferred to x.

Revealed Preference

Theorem (Afriat 1967)

Let be $\mathcal{D} = \{(x_t, p_t)\}_{t=1,...,T}$ be a collection of chosen bundles x_t at prices p_t. The following statements are equivalent

(i) The dataset can be rationalised by a locally nonsatiated preference relation $\succsim$ that admits a utility representation.

(ii) There is a continuous, concave, piecewise linear, strictly monotone utility function u that rationalises the dataset.

(iii) The dataset satisfies GARP.

(iv) There exist positive $\{u_t, \lambda_t\}_{t\in[T]}$ such that $u_s \leq u_t + \lambda_t p_t \cdot (x_s - x_t)$, for all $t, s = 1, ..., T$.

Intuition:

- (i) and (ii): with finite data LNS indistinguishable from (continuity, concavity, piecewise linearity, and strict monotonicity); the latter pose no additional constraints on the (finite) data.
- GARP (appropriately redefined) as the exact condition needed to rationalise data.
- (iv) far easier to check than GARP: reduces problem to simple linear programming.

Revealed Preference

Theorem (Afriat 1967)

Let be $\mathcal{D} = \{(x_t, p_t)\}_{t=1,\dots,T}$ be a collection of chosen bundles x_t at prices p_t. The following statements are equivalent

(ii) There is a continuous, concave, piecewise linear, strictly monotone utility function u that rationalises the dataset.

(iv) There exist positive $\{u_t, \lambda_t\}_{t\in[T]}$ such that $u_s \leq u_t + \lambda_t p_t \cdot (x_s - x_t)$, for all $t, s = 1, ..., T$.

Intuition:

- (iv) far easier to check than GARP: reduces problem to simple linear programming.
  - If u concave, then supergradients always exist, and, as u is differentiable almost everywhere (by concavity), $\partial u(x) = \{u'(x)\}$ almost everywhere.
  - (a.e.) $\forall x_s$, $u(x_s) \leq u(x_t) + u'(x_t) \cdot (x_s - x_t)$ (supergradient).
  - Suppose u indeed differentiable. Langragian for UMP is $u(x) + \lambda \cdot (w - p \cdot x)$. FOC: $u'(x) = \lambda p$.
  - Supergradient: $\forall q_t \in \partial u(x_t)$ and $\forall x_s$, $u(x_s) \leq u(x_t) + q_t \cdot (x_s - x_t)$.
  - Supergradient for differentiable function + FOC: $q_t = u'(x_t) = \lambda_t p_t$ and $\forall x_s$, $u(x_s) \leq u(x_t) + q_t \cdot (x_s - x_t) = u(x_t) + \lambda_t p_t \cdot (x_s - x_t)$.

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More

- Demand with Stochastic Choice: Abaluck & Adams-Prassl (2021 QJE).
- Revealed Preference with Measurement Error: Aguiar & Kashaev (2021 RES).
- Measuring Choice Inconsistency: Ok & Tserenjigmid (2022 TE), Ribeiro (2024 WP).
- Testing models with limited data: de Clippel & Rozen (2021 TE)