4. Monotone Comparative Statics of Individual Choices

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Overview

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2. General Definitions
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Some Motivation

Model of behaviour: $X(S; f) := \arg\max_{x \in S} f(x)$

f: incentives, what motivates a particular kind of actions, environment

S: constraints, rules of the environment, legal system, etc.

Comparative Statics: how changes in environment — in incentives f and/or constraints S — translate into changes in behaviour $X(S; f)$

Examples:

- Firm's input demand goes down when input's price increases
- When interest rate goes up new loans go down (and university enrollment also?)

In general too unruly $\rightarrow$ ***Monotone* Comparative Statics**

'increasing' S and/or f 'increases' X

- Fine idea, but what is a higher maximiser when the alternatives are not real numbers?
  And what does a higher *set* of maximisers (or feasible set) mean? And what is an increase in the *function* f?

Overview

Ordering Elements

Throughout: $(X, \geq)$ **partially ordered set**

$\geq$ is binary relation on X which is reflexive, transitive, and anti-symmetric.

Join and Meet

- **Join** of $x, x' \in X$ taken wrt X: $x \vee_X x' := \inf\{y \in X : y \geq x \text{ and } y \geq x'\}$.

  $x \vee_X x'$: $\geq$-smallest element in X which is larger than both x and x'.
- **Meet** of $x, x' \in X$ taken wrt X: $x \wedge_X x' := \sup\{y \in X : x \geq y \text{ and } x' \geq y\}$.

  $x \wedge_X x'$: $\geq$-largest element in X which is smaller than both x and x'.

  NB: inf and sup taken wrt $\geq$.

Lattice

Definition

(i) $(X, \geq)$ is a **lattice** iff it is a partially ordered set s.t. $\forall x, x' \in X$, $x \vee_X x' \in X$ and $x \wedge_X x' \in X$.
(Joins and meets of elements in X exist in X.)

(ii) $(X, \geq)$ is a **complete lattice** iff it is a lattice s.t. $\forall S \subseteq X$, $\vee_X S \equiv \sup_X S \in X$ and $\wedge_X S \equiv \inf_X S \in X$.
(Any subset of X attains its supremum and infimum in the set.)

(iii) $S \subseteq X$ is a **sublattice** of a partially ordered set $(X, \geq)$ iff $\forall x, x' \in S, x \vee_X x' \in S$ and $x \wedge_X x' \in S$.
(Joins and meets of elements in S taken in X wrt $\geq$ exist in S.)

(iv) $S \subseteq X$ is a **complete sublattice** of a partially ordered set $(X, \geq)$ iff it is a sublattice of $(X, \geq)$ and, $\forall S' \subseteq S$, $\sup_X S' \in S$ and $\inf_X S' \in S$.
(Any subset of S attains its supremum and infimum taken in X wrt $\geq$ in S.)

Lattice

Examples

(1) $((0,1), \geq)$ and $\geq$ natural order, is a lattice, but not a complete lattice.

(2) $(\mathbb{R}^k, \geq)$ and $\geq$ natural product order is lattice.
$\forall$ sublattice $S \subseteq \mathbb{R}^k$, S is complete sublattice if and only if it is compact.
Further: $(S, \geq)$ is then also a complete lattice.

(3) $(0,1) \subseteq \mathbb{R}$ is sublattice of $(\mathbb{R}, \geq)$ with $\geq$ natural order, but not a complete sublattice.

(4) Wrt natural product order, $\{(0,0), (1,0), (0,1), (2,2)\}$ is complete lattice, but not sublattice of $\mathbb{N}^2$.

(5) Wrt natural product order, $\{(0,0), (1,0), (0,1), (2,1), (1,2), (2,2)\}$ is not a lattice.

Ordering Sets

Unclear how to order sets based on partial order $\geq$;
there isn't an unequivocally 'right' way to do so.

Examples:

(1) $S = \{0\}$ and $S' = \{2\}$

(2) $S = \{0\}$ and $S' = \{1, 2\}$

(3) $S = \{0\}$ and $S' = \{-1, 2\}$

(4) $S = [0, 1]$ and $S' = [1, 2]$

(5) $S = [0, 1] \times [0, 1]$ and $S' = [1, 2] \times [1, 2]$

(6) $S = [0, 1] \times [0, 1]$ and $S' = [0, 2] \times [0, 2]$

Ordering Sets

Definition (Strong Set Order)

S' **strong set dominates** S (writing $S' \geq_{ss} S$) if $\forall x' \in S'$, $x \in S$, $x \vee x' \in S'$ and $x \wedge x' \in S$.

S' strong set dominates S if, taking any one element from each set, the join belongs to the dominating set and the meet to the dominated set.

In some cases, the strong set order can be too demanding.

Ordering Functions

Definition

Let $f : X \times T \to \mathbb{R}$, where X, T are partially ordered sets, and joins and meets of elements in $X \times T$ are wrt product order.

(i) f satisfies the **single-crossing property** (SCP) in $(x;t)$ if $\forall x, x' \in X$, $t, t' \in T$, s.t. $x' > x$ and $t' > t$, $f(x';t) - f(x;t) \geq (>)0 \implies f(x';t') - f(x;t') \geq (>)0$.
It satisfies the ***strict* single-crossing property** if last inequality is strict.

(ii) f has **increasing differences** (ID) in $(x;t)$ if $\forall x, x' \in X$, $t, t' \in T$, s.t. $x' > x$ and $t' > t$, $f(x';t') - f(x;t') \geq f(x';t) - f(x;t)$. It has ***strict* increasing differences** if last inequality is strict.

(iii) f is **quasisupermodular** (QSM) in (x, t) if $\forall y, y' \in X \times T$, $f(y) - f(y \wedge y') \geq (>)0 \implies f(y \vee y') - f(y') \geq (>)0$.

(iv) f is **supermodular** (SM) in (x, t) if $\forall y, y' \in X \times T$, $f(y \vee y') - f(y') \geq f(y) - f(y \wedge y')$.
f is **submodular** if $-f$ is supermodular.

Ordering Functions

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(iv) f is **supermodular** (SM) in (x,t) if $\forall y, y' \in X \times T$, $f(y \vee y') - f(y') \geq f(y) - f(y \wedge y')$.

NB: SM $\implies$ {QSM, ID} $\implies$ SCP.

SCP and QSM provide *ordinal* conditions on f, readily translatable into restrictions on preference relations. ID and SM are respective cardinal counterparts.

Behavioural Implications: With finite data, preference relations on a lattice have (a) a weakly monotone and quasisupermodular utility representation if and only if they have (b) a weakly monotone and supermodular utility representation (Chambers & Echenique 2009 JET)

Ordering Functions

Proposition

(i) If f and g are supermodular real-valued functions on X, then $\alpha f + \beta g$ are supermodular $\forall \alpha, \beta \geq 0$.

(ii) If $\exists$ strictly increasing $g : \mathbb{R} \to \mathbb{R}$ such that $g \circ f$ is supermodular, then f is quasisupermodular.

(iii) If $f \in \mathcal{C}^2$ in $y \in Y \equiv X \times T$, then f is supermodular in y if and only if $\frac{\partial^2}{\partial y_i \partial y_j} f \geq 0$, $\forall i \neq j$.

(iv) If X and Y are partially ordered sets, $X \times Y$ is a lattice with respect to the product order, and $f : X \times Y \to \mathbb{R}$ is supermodular, then $g(x) := \sup_{y \in Y} f(x, y)$ is supermodular.

(Proof left as an exercise.)

Ordering Functions

SM, QSM, ID, SCP are properties of *a* function.

But, we can think of $f(x,t)$ as defining a family of functions on X parametrized by t, $f_t(x) := f(x,t)$.

We can adjust definitions to order functions!

Definition

Let v, u be two real-valued functions on X; v **single-crossing** dominates u ($v \geq_{SC} u$) if $\forall x, x' \in X$ such that $x' \geq x$, $u(x') - u(x) \geq (>) \mathbf{0} \implies v(x') - v(x) \geq (>) \mathbf{0}$.

Overview

Strong Monotone Comparative Statics

Theorem (Milgrom & Shannon 1994 Theorem 4)

Let X be a lattice and v, u be two real-valued functions on X. v and u are quasisupermodular and v single-crossing dominates u if and only if, for $S' \geq_{ss} S$, $X(S'; v) \geq_{ss} X(S; u)$.

A very powerful result!

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Proof

$\Longrightarrow$: Take any $x \in X(S; u), x' \in X(S'; v)$.

As $S' \geq_{ss} S$, we have $x \wedge x' \in S$ and $x \vee x' \in S'$.

Then

$$\begin{aligned}
& x \in X(S; u) \\
\Longrightarrow\ & u(x) - u(x \wedge x') \geq 0 && \text{optimality of } x \\
\Longrightarrow\ & u(x \vee x') - u(x') \geq 0 && \text{quasisupermodularity of } u \\
\Longrightarrow\ & v(x \vee x') - v(x') \geq 0 && v \geq_{SC} u \\
\Longrightarrow\ & x \vee x' \in X(S'; v) && \text{optimality of } x';
\end{aligned}$$

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Proof

$\Longrightarrow$: Take any $x \in X(S; u), x' \in X(S'; v)$.

As $S' \geq_{SS} S$, we have $x \wedge x' \in S$ and $x \vee x' \in S'$.

Then $x \in X(S; u) \implies x \vee x' \in X(S'; v)$

and

$$\begin{aligned} & x' \in X(S'; v) \\ \implies & v(x \vee x') - v(x') \leq 0 && \text{optimality of } x' \\ \implies & v(x) - v(x \wedge x') \leq 0 && \text{quasisupermodularity of } v \\ \implies & u(x) - u(x \wedge x') \leq 0 && v \geq_{SC} u \\ \implies & x \wedge x' \in X(S; u) && \text{optimality of } x. \end{aligned}$$

Hence $X(S'; v) \geq_{SS} X(S; u)$.

Strong Monotone Comparative Statics

Theorem (Milgrom & Shannon 1994 Theorem 4)

Let X be a lattice and $v, u : X \to \mathbb{R}$. v and u are quasisupermodular and v single-crossing dominates u if and only if, for $S' \geq_{ss} S$, $X(S'; v) \geq_{ss} X(S; u)$.

Proof

$\Longleftarrow$:

- WTS necessity of quasisupermodularity.
  Let $S = \{x, x \wedge x'\}$, $S' = \{x', x \vee x'\}$, $\neg(x' \geq x)$, and $u = v$.

  Immediately, $S' \geq_{ss} S$.

  If we have $u(x) \geq (>) u(x \wedge x') \iff x \in (=) X(S; u)$.

  As $X(S'; u) \geq_{ss} X(S; u)$, then $x \in (=) X(S; u) \implies x \vee x' \in (=) X(S'; u) \implies u(x \vee x') \geq (>) u(x')$.
- WTS necessity of single-crossing.
  Let $S = \{x, x'\}$ with $x' > x$.

  As $X(S; v) \geq_{ss} X(S; u)$, $x' \in (=) X(S; u) \implies x' \in (=) X(S; v)$.

  And then, $u(x') - u(x) \geq (>) 0 \implies v(x') - v(x) \geq (>) 0$. $\square$

Strong Monotone Comparative Statics

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Let X be a lattice and $v, u : X \to \mathbb{R}$. v and u are quasisupermodular and v single-crossing dominates u if and only if, for $S' \geq_{ss} S$, $X(S'; v) \geq_{ss} X(S; u)$.

Two simple corollaries:

Corollary 1 (Milgrom & Shannon 1994)

Let X be a lattice and $f : X \to \mathbb{R}$. f is quasisupermodular if and only if, for $S' \geq_{ss} S$, $X(S'; f) \geq_{ss} X(S; f)$.

Proof

Take $v = u = f$. □

Strong Monotone Comparative Statics

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Corollary 2 (Milgrom & Shannon 1994)

Let X be a lattice, S a sublattice, and $f : X \to \mathbb{R}$. If f is quasisupermodular, then $X(S; f)$ is a sublattice of S.

Proof

Take $v = u = f$ and $S = S'$. Hence $X(S; f) \geq_{ss} X(S; f)$. $\square$

Strong Monotone Comparative Statics

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Theorem (Milgrom & Shannon 1994 Theorem 4')

Let X be a lattice and $v, u : X \to \mathbb{R}$. v and u are quasisupermodular and v strictly single-crossing dominates u then $\forall x' \in X(S'; v), x \in X(S; u), x' \geq x$.

any maximizer in $X(S'; v)$ is greater than *any* maximizer in $X(S; u)$!

Overview

More

- Comparative statics in choice and risk and uncertainty: Athey (2002 QJE).
- Comparative statics of equilibrium outcomes: more later.
- Applications: *Macro:* Effects of changes in consumers' preferences on prices and output (Acemoglu & Jensen 2015 JPE). *Econometrics:* Nonparametric identification (Lazzati 2015 QE). *Health:* Junior doctors' residency programme matching (Agarwal 2015 AER)