5. Expected Utility

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Overview

1. Decisions under Risk
2. Setup
3. Expected Utility
4. More

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Non-deterministic Outcomes

Until now: ignored whether or no DM knows exactly the consequences associated to their actions/choices

- **Buying as choosing a lottery**

  Computer may or may not be faulty

  Quality control tries to ensure things are fine, but faulty devices exist

  Ex-ante, one may know how likely a computer is to be faulty

  Different brands will have different fault probabilities

Risk: situations in which probabilities over outcomes are *known* and *objective*

(Later: uncertainty, when DM behaves as if according to subjective probability distribution)

Main questions for today:

(i) obtaining a tractable utility representation of preferences over lotteries (expected utility)

(ii) understanding what EU entails behaviourally and when it is more likely to be a better/worse description of behaviour

Overview

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Setup

- **Outcome space:** X, finite

  $x \in X$ entails a complete description of all relevant aspects of the environment
- **Probability measures on X:** $\Delta(X) := \{p : X \to [0,1] \mid \sum_x p(x) = 1\}$; (endowed with Euclidean metric)
- **Lottery:** $p \in \Delta(X)$ (i.e., a prob. distrib. on X)

  Can also think of p as vector in subset of $[0,1]^{|X|}$
- **Preference relation:** $\succsim \subseteq \Delta(X)^2$

Setup

- **Degenerate lottery/prob.:** $\delta_x \in \Delta(X) : \delta_x(x') = 1_{\{x'=x\}}$ ($1_{(\cdot)}$ is indicator function)
- **Probability mixture:** for $\alpha \in [0,1]$ and $p, p' \in \Delta(X)$,
  $\alpha p + (1-\alpha)p' \in \Delta(X)$ denotes lottery s.t.
  $$(\alpha p + (1-\alpha)p')(x) = \alpha p(x) + (1-\alpha)p'(x) \quad \forall x \in X$$
  Note:
  (1) $\Delta(X)$ convex wrt mixtures
  (2) Prob. mixture **is not** a compound lottery/prob. distr. over $\Delta(X)$

Overview

Preferences over Lotteries

If $\succsim$ continuous, then $\exists U : \Delta(X) \to \mathbb{R}$ s.t. $p \succsim p' \iff U(p) \geq U(p')$.

Suppose X is money and p an actual lottery.

Expected value representation: $U(x) = \mathbb{E}_p[x] \equiv \sum_x p(x)x$.

How general would such preferences be?

Choose δ_0 vs p : gain £10 wp 1/2 and lose £10 wp 1/2.

Choose δ_0 vs p' : gain £$10,000$ wp 1/2 and lose £$10,000$ wp 1/2.

Same expected value, but some people will choose p over £0 and p' over £0.

Consider p'' : gain £$10,000$ wp 1/2 and lose £10 wp 1/2.

p'' has far better upside than p and less bad downside than p'; reasonable to expect people to choose p'' over p or p'

Looking for representation that relaxes expected value assumption but retains tractability: separate probability and outcomes

Expected Utility

Definition

$\succsim$ on $\Delta(X)$ has an **expected utility (EU) representation** iff $\exists u : X \to \mathbb{R}$ such that $\forall p, p' \in \Delta(X)$, $p \succsim p' \iff \mathbb{E}_p[u] \geq \mathbb{E}_{p'}[u]$.

u: *Bernoulli* or *von Neumann–Morgenstern utility*

$\mathbb{E}_p[u] \equiv \sum_{x \in X} p(x)u(x)$

Continuity of $\succsim$ not sufficient for it to admit EU representation

Independence

Definition

Preference relation $\succsim$ on $\Delta(X)$ sat. **independence** if $\forall p, p' \in \Delta(X)$, $p \succsim (\succ)\, p'$ if and only if $\forall p'' \in \Delta(X)$, and $\forall \alpha \in (0,1]$, $\alpha p + (1-\alpha)p'' \succsim (\succ)\, \alpha p' + (1-\alpha)p''$.

NB: independence 'buys' linearity in probability
e.g., $p \sim p' \implies \alpha p + (1-\alpha)p' \sim p'$.

Independence **necessary** for EU representation $\because$ *expectations are linear in probabilities*

$\mathbb{E}_p[u] = (>)\, \mathbb{E}_{p'}[u] \implies \mathbb{E}_{\alpha p + (1-\alpha)p'}[u] = (>)\, \mathbb{E}_{p'}[u]$ (for $\alpha \in (0,1]$)

Immediately implies ruling out strict preference for randomisation, i.e., cannot have
$p \sim p'$ and $\alpha p + (1-\alpha)p' \succ p'$.

Continuity

Definition

Preference relation $\succsim$ on $\Delta(X)$ sat.

(i) **Archimedean property** if $\forall p, p', p'' \in \Delta(X)$ s.t. $p \succ p' \succ p''$, $\exists \alpha, \beta \in (0,1)$: $\alpha p + (1-\alpha)p'' \succ p' \succ \beta p + (1-\beta)p''$;

(ii) **vNM continuity** if $\forall p, p', p'' \in \Delta(X)$ s.t. $p \succsim p' \succsim p''$, $\exists \gamma \in [0,1] : \gamma p + (1-\gamma)p'' \sim p'$.

vNM continuity also **necessary** for EU representation:

if $\mathbb{E}_p[u] \geq \mathbb{E}_{p'}[u] \geq \mathbb{E}_{p''}[u]$, then $\exists \gamma \in [0,1]$ s.t. $\gamma \mathbb{E}_p[u] + (1-\gamma)\mathbb{E}_{p''}[u] = \mathbb{E}_{p'}[u]$;

and linearity $\mathbb{E}_p$ wrt p implies

$$\gamma \mathbb{E}_p[u] + (1-\gamma)\mathbb{E}_{p''}[u] = \mathbb{E}_{\gamma p + (1-\gamma)p''}[u].$$

EU Representation

Theorem (von Neumann & Morgenstern 1953)

Let X be finite and $\succsim$ a preference relation on $\Delta(X)$.

(i) $\succsim$ satisfies independence and vNM continuity if and only if it admits an expected utility representation u.

(ii) If u and v are two expected utility representations of $\succsim$, then $\exists \alpha > 0, \beta \in \mathbb{R}$ such that $v = \alpha u + \beta$.

EU Representation

Proof

If part of (i) already discussed. Focus on only if.

Step 1. $\exists \delta_{\overline{x}}, \delta_{\underline{x}} \in \Delta(X)$ such that $\forall \delta_x \in \Delta(X)$, $\delta_{\overline{x}} \succsim \delta_x \succsim \delta_{\underline{x}}$.

Step 2. $\forall p, p' \in \Delta(X)$ s.t. $p \succsim p'$, $\forall \{p_i\}_{i=1,\ldots,n} \subseteq \Delta(X)$, and $\{\alpha_i\}_{i=0,\ldots,n} \subset [0,1] : \sum_{i=0}^{n} \alpha_i = 1$, we have

$$\alpha_0 p + \sum_{i \in [n]} \alpha_i p_i \succsim \alpha_0 p' + \sum_{i \in [n]} \alpha_i p_i.$$

Proof:

(i) If $\alpha_0 \in \{0,1\}$, claim trivially true.

(ii) For $\alpha_0 \in (0,1)$, $1 - \alpha_0 = \sum_{i \in [n]} \alpha_i$, define $p'' := \sum_{i \in [n]} \frac{\alpha_i}{1-\alpha_0} p_i$ ($\in \Delta(X) \because$ convexity).

(iii) By independence,

$$\alpha_0 p + \sum_{i \in [n]} \alpha_i p_i = \alpha_0 p + (1 - \alpha_0) p''$$

$$\succsim \alpha_0 p' + (1 - \alpha_0) p'' = \alpha_0 p' + \sum_{i \in [n]} \alpha_i p_i.$$

EU Representation

Proof: (i) 'only if'

Step 3. $\forall p \in \Delta(X)$, $\delta_{\overline{x}} \succsim p \succsim \delta_{\underline{x}}$.

Proof:

Fix an order on $X = \{x_1, x_2, ..., x_n\}$ s.t. $x_1 = \overline{x}$ and $x_n = \underline{x}$. By Step 1 and repeated application of Step 2,

$$
\begin{aligned}
\delta_{\overline{x}} &= p(x_1)\delta_{\overline{x}} + p(x_2)\delta_{\overline{x}} + \cdots + p(x_n)\delta_{\overline{x}} \\
&\succsim p(x_1)\delta_{x_1} + p(x_2)\delta_{\overline{x}} + \cdots + p(x_n)\delta_{\overline{x}} \\
&\succsim p(x_1)\delta_{x_1} + p(x_2)\delta_{x_2} + \cdots + p(x_n)\delta_{\overline{x}} \\
&\succsim \cdots \\
&\succsim p(x_1)\delta_{x_1} + p(x_2)\delta_{x_2} + \cdots + p(x_n)\delta_{x_n} = p \\
&\succsim \cdots \\
&\succsim \delta_{\underline{x}}
\end{aligned}
$$

If $\delta_{\overline{x}} \sim \delta_{\underline{x}}$, set $u = c$ constant; done! (why?)

Otherwise, it must be that $\delta_{\overline{x}} \succ \delta_{\underline{x}}$.

EU Representation

Proof: (i) 'only if'

Step 3. $\forall p \in \Delta(X), \delta_{\overline{x}} \succsim p \succsim \delta_{\underline{x}}$.

Step 4. $\forall \alpha, \beta : 1 \geq \alpha > \beta \geq 0, \quad \alpha\delta_{\overline{x}} + (1-\alpha)\delta_{\underline{x}} \succ \beta\delta_{\overline{x}} + (1-\beta)\delta_{\underline{x}}$.

Proof:

(i) By independence,

$$\left(\frac{\alpha-\beta}{1-\beta}\right)\delta_{\overline{x}} + \left[1-\left(\frac{\alpha-\beta}{1-\beta}\right)\right]\delta_{\underline{x}} \succ \left(\frac{\alpha-\beta}{1-\beta}\right)\delta_{\underline{x}} + \left[1-\left(\frac{\alpha-\beta}{1-\beta}\right)\right]\delta_{\underline{x}} = \delta_{\underline{x}}.$$

(ii) Again by independence,

$$\alpha\delta_{\overline{x}} + (1-\alpha)\delta_{\underline{x}} = \beta\delta_{\overline{x}} + (1-\beta)\left[\left(\frac{\alpha-\beta}{1-\beta}\right)\delta_{\overline{x}} + \left[1-\left(\frac{\alpha-\beta}{1-\beta}\right)\right]\delta_{\underline{x}}\right]$$

$$\succ \beta\delta_{\overline{x}} + (1-\beta)\left[\left(\frac{\alpha-\beta}{1-\beta}\right)\delta_{\underline{x}} + \left[1-\left(\frac{\alpha-\beta}{1-\beta}\right)\right]\delta_{\underline{x}}\right] = \beta\delta_{\overline{x}} + (1-\beta)\delta_{\underline{x}}$$

EU Representation

Proof: (i) 'only if'

Step 3. $\forall p \in \Delta(X)$, $\delta_{\overline{x}} \succsim p \succsim \delta_{\underline{x}}$.

Step 4. $\forall \alpha, \beta : 1 \geq \alpha > \beta \geq 0, \quad \alpha\delta_{\overline{x}} + (1-\alpha)\delta_{\underline{x}} \succ \beta\delta_{\overline{x}} + (1-\beta)\delta_{\underline{x}}$.

Step 5. $\forall p \in \Delta(X)$, $\exists! \gamma(p) \in [0,1] : \gamma(p)\delta_{\overline{x}} + (1-\gamma(p))\delta_{\underline{x}} \sim p$.

Proof:

(i) By Step 3, $\delta_{\overline{x}} \succsim p \succsim \delta_{\underline{x}}$.

(ii) vNM continuity ensures existence of a $\gamma \in [0,1]$.

(iii) By Step 4, it must be unique (why?).

EU Representation

Proof: (i) 'only if'

Step 3. $\forall p \in \Delta(X)$, $\delta_{\overline{x}} \succsim p \succsim \delta_{\underline{x}}$.

Step 4. $\forall \alpha, \beta : 1 \geq \alpha > \beta \geq 0$, $\quad \alpha\delta_{\overline{x}} + (1-\alpha)\delta_{\underline{x}} \succ \beta\delta_{\overline{x}} + (1-\beta)\delta_{\underline{x}}$.

Step 5. $\forall p \in \Delta(X)$, $\exists! \gamma(p) \in [0,1] : \gamma(p)\delta_{\overline{x}} + (1-\gamma(p))\delta_{\underline{x}} \sim p$.

Step 6. Define $u : X \to \mathbb{R}$ s.t. $u(x) = \gamma(\delta_x)$. Then, $\gamma(p) = \sum_{i\in[n]} p(x_i)\gamma(\delta_{x_i})$.

Proof: WTS

$$p \sim \left(\sum_{i\in[n]} p(x_i)\gamma(\delta_{x_i})\right)\delta_{\overline{x}} + \left(1 - \sum_{i\in[n]} p(x_i)\gamma(\delta_{x_i})\right)\delta_{\underline{x}}.$$

By repeated application of independence, Step 2, and definition of γ,

$$p = \sum_{i=1}^{n} p(x_i)\delta_{x_i} \sim \sum_{i=1}^{n} p(x_i)\left[\gamma(\delta_{x_i})\delta_{\overline{x}} + (1-\gamma(\delta_{x_i}))\delta_{\underline{x}}\right]$$

$$= \sum_{i=1}^{n} p(x_i)\left(\gamma(\delta_{x_i})\right)\delta_{\overline{x}} + \sum_{i=1}^{n} p(x_i)\left((1-\gamma(\delta_{x_i}))\right)\delta_{\underline{x}}$$

(i) The claim follows from Step 5.

EU Representation

Proof: (i) 'only if'

Step 3. $\forall p \in \Delta(X)$, $\delta_{\overline{x}} \succsim p \succsim \delta_{\underline{x}}$.

Step 4. $\forall \alpha, \beta : 1 \geq \alpha > \beta \geq 0, \quad \alpha\delta_{\overline{x}} + (1-\alpha)\delta_{\underline{x}} \succ \beta\delta_{\overline{x}} + (1-\beta)\delta_{\underline{x}}$.

Step 5. $\forall p \in \Delta(X), \exists! \gamma(p) \in [0,1] : \gamma(p)\delta_{\overline{x}} + (1-\gamma(p))\delta_{\underline{x}} \sim p$.

Step 6. Define $u : X \to \mathbb{R}$ s.t. $u(x) = \gamma(\delta_x)$. Then, $\gamma(p) = \sum_{i\in[n]} p(x_i)\gamma(\delta_{x_i})$.

Step 7. Take any $p, p' \in \Delta(X)$. $p \succsim p' \iff \mathbb{E}_p[u] \geq \mathbb{E}_{p'}[u]$.

Proof:

(i) By Step 4 and Step 5, $\gamma(p)\delta_{\overline{x}} + (1-\gamma(p))\delta_{\underline{x}} \sim p \succsim p' \sim \gamma(p')\delta_{\overline{x}} + (1-\gamma(p'))\delta_{\underline{x}}$, iff $\gamma(p) \geq \gamma(p')$.

(ii) By Step 5 and Step 6, it follows $\mathbb{E}_p[\gamma] = \sum_{i\in[n]} p(x_i)\gamma(\delta_{x_i}) = \gamma(p)$.

(iii) By definition, $\mathbb{E}_p[u] = \mathbb{E}_p[\gamma]$.

EU Representation

Proof: (ii)

WTS: If u and v are two EU representations of $\succsim$, then $\exists \alpha > 0, \beta \in \mathbb{R}$ s.t. $v = \alpha u + \beta$.

(i) If $\delta_{\overline{x}} \sim \delta_{\underline{x}}$, both u and v are constants; done. Then, let $\delta_{\overline{x}} \succ \delta_{\underline{x}}$.

(ii) Take u defined in (i) and let v be some other EU representation of $\succsim$.

(iii) Note that $\forall p \in \Delta(X)$, it must $v(\overline{x}) \geq \mathbb{E}_p[v] \geq v(\underline{x})$.

(iv) Define $\phi(p) \in [0,1] : \phi(p)v(\overline{x}) + (1 - \phi(p))v(\underline{x}) = \mathbb{E}_p[v]$.
There is exactly one such number.

(v) Since
$$\phi(p)v(\overline{x}) + (1 - \phi(p))v(\underline{x}) = \mathbb{E}_{\phi(p)\delta_{\overline{x}} + (1-\phi(p))\delta_{\underline{p}}}[v],$$
we have that
$$\phi(p)\delta_{\overline{x}} + (1 - \phi(p))\delta_{\underline{x}} \sim p \sim \gamma(p)\delta_{\overline{x}} + (1 - \gamma(p))\delta_{\underline{x}}.$$

(vi) By Step 5, $\gamma(p) = \phi(p)$. Hence, $v(x_i) = \gamma(\delta_{x_i})v(\overline{x}) + (1 - \gamma(\delta_{x_i}))v(\underline{x})$.

(vii) Hence, $u = \frac{v - v(\underline{x})}{v(\overline{x}) - v(\underline{x})} \implies v = \alpha u + \beta$, with $\alpha = v(\overline{x}) - v(\underline{x})$ and $\beta = v(\underline{x})$. □

EU Representation

Theorem (von Neumann & Morgenstern 1953)

Let X be finite and $\succsim$ a preference relation on $\Delta(X)$.

(i) $\succsim$ satisfies independence and vNM continuity if and only if it admits an expected utility representation u.

(ii) If u and v are two expected utility representations of $\succsim$, then $\exists \alpha > 0, \beta \in \mathbb{R}$ such that $v = \alpha u + \beta$.

EU representations are unique up to positive affine transformations; cardinal interpretation of u.

Overview

Compound Lotteries

- What is a compound lottery?
  - p: +£5 wp 1/2, -£5 wp 1/2
  - p': +£$5,000$ wp 1/2, -£5 wp 1/2
  - ℓ: p wp 1/2, p' wp 1/2
  - $\ell \neq p''$: +£$5,000$ wp 1/4, +£5 wp 1/4, -£5 wp 1/2
- Segal (1990 Ecta) discusses preferences $\trianglerighteq$ on $\Delta(\Delta(X))$ and the relation with preferences $\succsim$ on $\Delta(X)$
  - Treating compound lotteries and mixtures differently: failure to reduce compound Lotteries
  - Turns out that attitudes specific to compound lotteries seem to be closer related to attitudes toward uncertainty than to attitudes toward simple lotteries (e.g., Ortoleva & Dean 2019 PNAS)

Allais Paradox (1953 Ecta)

Paris, sometime between 12 and 17 May 1952, over lunch at conference on choice under risk

Maurice Allais asks J. Leonard Savage

1. Which of the following two gambles do you prefer?
   a) £2 million wp 1; or
   b) £2 million wp .89; £10 million wp .10; nothing wp .01.

   Savage chose a)

Allais asked:

2. Which of the following two gambles do you prefer?
   A) nothing wp .89; £2 million wp .11; or
   B) nothing wp .90; £10 million wp .10.

   Savage chose B)

Choosing a) and B) [or b) and A)] cannot be rationalised by EU (why?)

Issues with Expected Utility

Common consequence paradox.

- Also common ratio paradox (preference reversal following mixture with 0).

Should we just throw away EU?

- EU has **normative** appeal and people *should* behave according to its principles.
- (Savage considered he had been ‘tricked’ and wrote to Allais saying he still thought principles were sound)
- EU is still a useful model for choice under risk
  - Understanding better when it holds and when it fails is illuminating

More

Rank-Dependent Expected Utility (Quiggin, 1982 JEBO); cumulative prospect theory (Tversky & Kahneman, 1992 JRU)

- Main gist: small probabilities of the worst events loom larger than they are
- Attracted lots of discussion recently (a good topic for a survey)

Cautious Expected Utility (Cerreia-Vioglio, Dillenberger, & Ortoleva, 2015 Ecta)

- Relaxes independence to: $\forall p, p' \in \Delta(X)$, $x \in X$, and $\alpha \in [0, 1]$, if $p \succsim \delta_x$, then $\alpha p + (1 - \alpha)p' \succsim \alpha\delta_x + (1 - \alpha)p'$.

Ordered Reference Dependent Choice (Lim, 2021 WP)

- Way in which alternatives are compared depend on set of alternatives, e.g., existence of sure things, 'riskiness' of riskiest alternative, etc.

Cognitive Perception of Risk

- Choice under risk and computational complexity (Oprea, 2024 AER)
- Uncertainty regarding valuation

Robustness and Misspecification

- Climate change, limited knowledge, limited modelling capacity
- Variational preferences (Cerreia-Vioglio et al.)