6. Risk Attitudes

Duarte Gonçalves
University College London

MRes Microconomics

Overview

Understanding attitudes toward risk is fundamental to understand behaviour

- how people constitute their financial portfolios;
- behaviour in the context of a pandemic;
- purchasing decisions;
- willingness to take up a job or continue searching for a better one;
- voting for new parties/candidates; etc.

Focus on case of preferences over wealth

- Indirect utility: $v(p, w) = \max_{x \in B(p,w)} u(x)$; under some assumptions (which?) we get $v(p, \cdot)$ strictly increasing.
- With preferences over lotteries over wealth and some more assumptions, we get something like an EU representation: $\mathbb{E}_F[v(p, \cdot)]$

Overview

Today:

1. Introduce and study behavioural notions of risk aversion (which can be tested/falsified with data).
2. Provide a behavioural way to compare individuals in terms of their risk attitudes, even if not risk averse;
   Show how this relates to structural properties of their EU representations.
3. Examine implications (e.g., behavioural fingerprints) of patterns of how attitudes toward risk can be affected by wealth.

Overview

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4. Comparing Risk Attitudes
5. Risk Attitudes with Changing Wealth
6. Two Functional Forms for Expected Utility
7. More

Overview

Setup

- **Outcome space:** $X \subseteq \mathbb{R}$, convex

  $x \in X$: DM's final wealth.

- **Cumulative Probability Distributions Function** F

  $F : \mathbb{R} \to [0,1]$ s.t. F is nondecreasing, right-continuous, $\lim_{x \to -\infty} F(x) = 0$, and $\lim_{x \to \infty} F(x) = 1$ with support on X, i.e. $\mathbb{P}_F(X) = \int_X dF(x) = 1$.

  Expectation Operator: $\mathbb{E}_F[\cdot]$

  Mean: $\mu_F = \mathbb{E}_F[x]$

- $\mathcal{F}$: set of (Borel) probability measures on X with finite mean μ_F (endowed with topology of weak convergence)

- **Preference Relation:** $\succsim \subseteq \mathcal{F}^2$ sat. independence, Archimedean property, continuity, and monotonicity ($x > y \implies \delta_x \succ \delta_y$)

- **EU Representation:** $u : X \to \mathbb{R}$ s.t. $\forall F, G, F \succsim G \iff \mathbb{E}_F[u] \geq \mathbb{E}_G[u]$

  Implies independence and Archimedean property

  (glossing over some details here — see section 5.2. in Kreps (2012))

  Define $U(F) := \mathbb{E}_F[u]$

Setup

Assumption

Preference relation $\succsim$ on $\mathcal{F}$ has EU representation $u : X \to \mathbb{R}$ strictly increasing.

Overview

Risk Attitudes

Risk attitudes: general patterns of behaviour toward risk

- Almost taxonomical approach
- Capture idea of avoiding/seeking risk
- **Risk aversion** as rejecting fair gambles ($\pm$£x wp 1/2)
- Extend idea to more general lotteries

Definition

A preference relation $\succsim$ on $\mathcal{F}$ is

(i) **risk averse** if $\forall F \in \mathcal{F}$, $\delta_{\mu_F} \succsim F$;

(ii) **risk neutral** if $\forall F \in \mathcal{F}$, $\delta_{\mu_F} \sim F$;

(iii) **risk seeking** if $\forall F \in \mathcal{F}$, $\delta_{\mu_F} \precsim F$.

Definition

(i) The **certainty equivalent** of F for $\succsim$ is $c(F, \succsim) \in X$ such that $\delta_{c(F,\succsim)} \sim F$.

(ii) The **risk premium** of F for $\succsim$ is the real number $R(F, \succsim) := \mu_F - c(F, \succsim)$.

Risk Attitudes

Theorem

The following statements are equivalent:

(i) $\succsim$ is risk averse (risk seeking).

(ii) $c(F, \succsim) \leq (\geq) \mu_F, \forall F \in \mathcal{F}$.

(iii) u is concave (convex).

Proof

(i) $\iff$ (ii): $\delta_{\mu_F} \succsim F \iff u(\mu_F) = U(\delta_{\mu_F}) \geq U(F) = u(c(F, \succsim))$ (using monotonicity of u).

(i) $\implies$ (iii): $\forall x, x' \in X : x > x'$, and $\forall \alpha \in [0, 1]$, let F deliver x wp α and x' with wp $1 - \alpha$.
Then, $u(\alpha x + (1 - \alpha)x') = u(\mu_F) = U(\delta_{\mu_F}) \geq U(F) = \mathbb{E}_F[u] = \alpha u(x) + (1 - \alpha)u(x')$.

(i) $\impliedby$ (iii): Take same F as defined. Then, $U(\delta_{\mu_F}) = u(\mu_F) \geq \mathbb{E}_F[u] = U(F)$.

The proof of equivalences for risk seeking preferences is symmetric. □

Overview

Comparing Risk Attitudes

A person may take a fair bet for low stakes, but not if stakes are too high

Risk averse? Risk seeking?

Risk averse is too demanding

Can we nevertheless compare different people's risk attitudes?

Comparing Risk Attitudes

Definition

$\succsim^a$ is said to be more risk averse than $\succsim^b$ if $F \succsim^a \delta_x \implies F \succsim^b \delta_x, \forall F \in \mathcal{F}, \forall x \in X$.

Whenever person b declines a bet in favour of some sure thing,
a more risk averse person a declines too

Definition

For an EU representation $u \in C^2$ and $x \in X$, define the **Arrow-Pratt coefficient of absolute risk aversion** as $r_A(x,u) := -\frac{u''(x)}{u'(x)}$.

Measures the rate at which mg utility of wealth changes

Why not just the curvature? (more/less concave)

Comparing Risk Attitudes

Theorem

Let $\succsim^a, \succsim^b$ be two preference relations on $\mathcal{F}$ and u^a, u^b be strictly increasing expected utility representations of $\succsim^a, \succsim^b$, respectively. The following statements are equivalent:

(i) $\succsim^a$ is more risk averse than $\succsim^b$.

(ii) $c(F, \succsim^a) \leq c(F, \succsim^b), \forall F \in \mathcal{F}$.

(iii) If $u^b \in \mathcal{C}^0$, then $\exists$ is a real-valued, strictly increasing, concave function ϕ such that $u^a = \phi \circ u^b$.

(iv) If $u^a, u^b \in \mathcal{C}^2$, then $r_A(x, u^a) \geq r_A(x, u^b)$ for any $x \in X$.

Comparing Risk Attitudes

Theorem

(i) $\succsim^a$ is more risk averse than $\succsim^b$.

(ii) $c(F, \succsim^a) \leq c(F, \succsim^b), \forall F \in \mathcal{F}$.

Proof

(i) $\iff$ (ii): $\delta_{c(F,\succsim^a)} \sim^a F \implies \delta_{c(F,\succsim^a)} \precsim^b F \sim^b \delta_{c(F,\succsim^b)} \iff c(F, \succsim^a) \leq c(F, \succsim^b)$.

Comparing Risk Attitudes

Theorem

(ii) $c(F, \succsim^a) \leq c(F, \succsim^b), \forall F \in \mathcal{F}$.

(iii) If $u^b \in \mathcal{C}^0$, then $\exists$ is a real-valued, strictly increasing, concave function ϕ such that $u^a = \phi \circ u^b$.

Proof

(ii) $\implies$ (iii): u^b strictly increasing $\implies$ ${u^b}^{-1}$ well-defined.

$\phi := u^a \circ {u^b}^{-1}$; strictly increasing $\because$ u^a, u^b strictly increasing.

X convex and u^b is continuous and strictly increasing $\implies$ $u^b(X)$ convex.

Further: $\phi(u^b(x)) = u^a({u^b}^{-1}(u^b(x))) = u^a(x)$.

We prove by contrapositive. Suppose ϕ not concave.

$\implies$ $\exists x, x' \in X$, and $\alpha \in (0,1) : \phi(\alpha u^b(x) + (1-\alpha)u^b(x')) < \alpha\phi(u^b(x)) + (1-\alpha)\phi(u^b(x'))$.

Comparing Risk Attitudes

Theorem

(ii) $c(F, \succsim^a) \leq c(F, \succsim^b), \forall F \in \mathcal{F}$.

(iii) If $u^b \in \mathcal{C}^0$, then $\exists$ is a real-valued, strictly increasing, concave function ϕ such that $u^a = \phi \circ u^b$.

Proof

(ii) $\implies$ (iii): $\phi := u^a \circ {u^b}^{-1}$; strictly increasing.

Suppose ϕ not concave.

$\implies \exists x, x' \in X$, and $\alpha \in (0,1) : \phi(\alpha u^b(x) + (1-\alpha)u^b(x')) < \alpha\phi(u^b(x)) + (1-\alpha)\phi(u^b(x'))$.

Let F yield x wp α and x' wp $1 - \alpha$.

Note: $\phi(\alpha u^b(x) + (1-\alpha)u^b(x')) = \phi(\mathbb{E}_F[u^b])$ and $\alpha\phi(u^b(x)) + (1-\alpha)\phi(u^b(x')) = \mathbb{E}_F[\phi \circ u^b]$.

$$\implies u^a(c(F, \succsim^a)) = U^a(F) = \mathbb{E}_F[u^a] = \mathbb{E}_F[\phi \circ u^b]$$

$$> \phi(\mathbb{E}_F[u^b]) = \phi(U^b(F)) = \phi(u^b(c(F, \succsim^b)) = u^a(c(F, \succsim^b)).$$

Monotonicity of $u^a \implies c(F, \succsim^a) > c(F, \succsim^b)$.

Comparing Risk Attitudes

Theorem

(ii) $c(F, \succsim^a) \leq c(F, \succsim^b), \forall F \in \mathcal{F}$.

(iii) If $u^b \in \mathcal{C}^{\mathbf{0}}$, then $\exists$ is a real-valued, strictly increasing, concave function ϕ such that $u^a = \phi \circ u^b$.

Proof

(ii) $\Longleftarrow$ (iii):

$$u^a(c(F, \succsim^a)) = U^a(F) = \mathbb{E}_F[u^a] = \mathbb{E}_F[\phi \circ u^b]$$

$$\leq \phi(\mathbb{E}_F[u^b]) = \phi(U^b(F)) = \phi(u^b(c(F, \succsim^b))) = u^a(c(F, \succsim^b)),$$

u^a strictly increasing $\implies c(F, \succsim^a) \leq c(F, \succsim^b)$.

Comparing Risk Attitudes

Theorem

(iii) If $u^b \in \mathcal{C}^0$, then $\exists$ is a real-valued, strictly increasing, concave function ϕ such that $u^a = \phi \circ u^b$.

(iv) If $u^a, u^b \in \mathcal{C}^2$, then $r_A(x, u^a) \geq r_A(x, u^b)$ for any $x \in X$.

Proof

(iii) $\iff$ (iv):

u^a, u^b strictly increasing and differentiable $\implies$ ${u^a}', {u^b}' > 0$.

$\phi := u^a \circ {u^b}^{-1}$ and $u^a, u^b \in \mathcal{C}^2 \implies \phi' > 0$ and $\phi \in \mathcal{C}^2$.

By definition, ${u^a}''(x) = \phi''(u^b(x))({u^b}'(x))^2 + \phi'(u^b(x)){u^b}''(x)$.

$$r_A(x, u^a) = -\frac{\phi''(u^b(x))({u^b}'(x))^2 + \phi'(u^b(x)){u^b}''(x)}{\phi'(u^b(x)){u^b}'(x)} = -\frac{\phi''(u^b(x)){u^b}'(x)}{\phi'(u^b(x))} - \frac{{u^b}''(x)}{{u^b}'(x)} \geq r_A(x, u^b)$$

$\iff \phi'' \leq 0$.

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Risk Attitudes with Changing Wealth

Folk wisdom: wealthier people are more risk seeking.

How can we formalise (and test) statements like these?

Answer: use our notion of comparative risk aversion + shift lotteries by baseline Wealth

Varying Wealth: For lottery F and $w \in \mathbb{R}$, write $F + w \in \mathcal{F}$ as lottery arising from adding w to every outcome, i.e., $(F + w)(x) := F(x - w)$.

Wealth-Dependent Preferences: For pref. rel. $\succsim$ on $\mathcal{F}$, write $\succsim_w$ as preference given *additional* wealth w: $F \succsim_w G \iff F + w \succsim G + w$.

EU: $u_w(x) := u(x + w)$ and $U_w(F) := \mathbb{E}_F[u_w]$.

Risk Attitudes with Changing Wealth

Definition

u exhibits **decreasing/constant/increasing absolute risk aversion** (DARA/CARA/IARA) if $r_A(x, u)$ is decreasing/constant/increasing in x.

Theorem

Let $\succsim$ be a preference relation on $\mathcal{F}$ and u a strictly increasing expected utility representation. The following statements are equivalent:

(i) If $u \in \mathcal{C}^2$, u exhibits DARA.

(ii) $\succsim_{w^a}$ is more risk averse than $\succsim_{w^b}$, $\forall w^a \leq w^b$.

(iii) $c(F, \succsim_{w^a}) \leq c(F, \succsim_{w^b})$, $\forall F \in \mathcal{F}, \forall w^a \leq w^b$.

(iv) $w^b - w^a \leq c(F + w^b, \succsim) - c(F + w^a, \succsim)$, $\forall F \in \mathcal{F}, \forall w^a \leq w^b$.

Risk Attitudes with Changing Wealth

Theorem

Let $\succsim$ be a preference relation on $\mathcal{F}$ and u a strictly increasing expected utility representation. The following statements are equivalent:

(i) If $u \in \mathcal{C}^2$, u exhibits DARA.

(ii) $\succsim_{w^a}$ is more risk averse than $\succsim_{w^b}$, $\forall w^a \leq w^b$.

(iii) $c(F, \succsim_{w^a}) \leq c(F, \succsim_{w^b})$, $\forall F \in \mathcal{F}, \forall w^a \leq w^b$.

(iv) $w^b - w^a \leq c(F + w^b, \succsim) - c(F + w^a, \succsim)$, $\forall F \in \mathcal{F}, \forall w^a \leq w^b$.

Proof

(i) $\iff$ (ii): Follows from (i) $\iff$ (iv) in previous theorem.

(ii) $\iff$ (iii): Follows from (i) $\iff$ (ii) in previous theorem.

(iii) $\iff$ (iv): Need an intermediate lemma:

Risk Attitudes with Changing Wealth

Theorem

(iii) $c(F, \succsim_{w^a}) \leq c(F, \succsim_{w^b}), \forall F \in \mathcal{F}, \forall w^a \leq w^b$.

(iv) $w^b - w^a \leq c(F + w^b, \succsim) - c(F + w^a, \succsim), \forall F \in \mathcal{F}, \forall w^a \leq w^b$.

Proof

(iii) $\Longleftrightarrow$ (iv): Need an intermediate lemma:

Lemma: Let $\succsim$ be preference relation on $\mathcal{F}$, and u a strictly increasing expected utility representation. Then, $c(F, \succsim_w) = c(F + w, \succsim) - w$.

Proof of the lemma:

$$u(c(F, \succsim_w) + w) = u_w(c(F, \succsim_w)) = \mathbb{E}_F[u_w] = \int_X u_w(x)dF(x) = \int_X u(x + w)dF(x)$$

$$= \int_{X+w} u(x)dF(x - w) = \mathbb{E}_{F+w}[u] = u(c(F + w, \succsim)),$$

where $X + w := \{x + w \mid x \in X\}$.

(iii) $\Longleftrightarrow$ (iv): $\mathbf{0} \leq c(F, \succsim_{w^b}) - c(F, \succsim_{w^a}) = c(F + w^b, \succsim) - w^b - c(F + w^a, \succsim) + w^a$. □

Overview

Constant Absolute Risk Aversion

Utility representations of CARA preferences are pinned-down
(of course, up to positive affine transformations).

Proposition

$\succsim$ exhibits CARA and admits a twice-differentiable utility representation u if and only if $\exists \alpha > 0, \beta \in \mathbb{R}$ such that $u(x) = -\alpha \text{sign}(\gamma) \exp(-\gamma x) + \beta$ if $\gamma \neq 0$, and $u(x) = \alpha x + \beta$ if otherwise, where $\gamma = r_A(x, u), \forall x \in X$.

Proof

$$r_A(x,u) = -\frac{u''(x)}{u'(x)} = \gamma \iff \int \gamma dx = -\int \frac{u''(x)}{u'(x)} dx \iff \ln u'(x) + k_1 = -\gamma x.$$

If $\gamma \neq 0$, then

$$\ln u'(x) + k_1 = -\gamma x \iff u'(x) = \exp(-\gamma x - k_1) \iff u(x) = -\frac{\exp(-k_1)}{\gamma} \exp(-\gamma x) + k_2,$$

for some $k_1, k_2 \in \mathbb{R}$. If instead $\gamma = 0$, $u''(x) = 0 \implies u(x) = \alpha x + \beta$. □

Constant Relative Risk Aversion

Definition

Let $u \in C^2$ be a EU representation of $\succsim$. The **Arrow-Pratt coefficient of relative risk aversion** at $x \in X$ is given by $r_R(x, u) := -\frac{u''(x)}{u'(x)}x$.

Proposition

$\succsim$ exhibits CRRA and admits a twice-differentiable utility representation u if and only if $\exists \alpha > 0, \beta \in \mathbb{R}$ such that $u(x) = \alpha \frac{x^{1-\gamma}}{1-\gamma} + \beta$, if $\gamma \neq 1$, and $u(x) = \alpha \ln(x) + \beta$ if otherwise, where $\gamma = r_R(x, u), \forall x \in X$.

Proof

$$r_R(x,u) = -\frac{u''(x)}{u'(x)}x = \gamma \iff \int \gamma \frac{1}{x} dx = -\int \frac{u''(x)}{u'(x)} dx \iff \ln u'(x) = -\gamma \ln x + k_1$$

$$\iff u'(x) = \exp(k_1)x^{-\gamma} \iff u(x) = \exp(k_1)\frac{x^{1-\gamma}}{1-\gamma} + k_2,$$

if $\gamma \neq 1$ for some $k_1, k_2 \in \mathbb{R}$. If $\gamma = 1$, then $u'(x) = \exp(k_1)x^{-1} \iff u(x) = \exp(k_1)\ln x + k_2$.

Constant Relative Risk Aversion

Definition

Let $u \in C^2$ be a EU representation of $\succsim$. The **Arrow-Pratt coefficient of relative risk aversion** at $x \in X$ is given by $r_R(x, u) := -\frac{u''(x)}{u'(x)}x$.

Interesting fact about CRRA preferences: *the only* class of utility functions that, in a Solow model with technological progress at rate g, delivers a balanced growth path.

Overview

More Issues with Expected Utility

Another issue: small-stakes risk aversion

Rabin's Calibration theorem (2000 Ecta):

- If u concave, changes in small stakes approx. linear
- Small-stakes risk aversion gives rise to wild estimates:
- If reject -$100 wp 1/2, +$125 wp 1/2 for wealth levels less than $300k, then reject -$600 wp 1/2, +$36B wp 1/2 for starting wealth of $290k

More Issues with Expected Utility

Other ways to risk aversion

Rank-Dependent Expected Utility (Quiggin, 1982 JEBO); cumulative prospect theory (Tversky & Kahneman, 1992 JRU)

- Main gist: small probabilities of the worst events loom larger than they are
- Attracted lots of discussion recently (a good topic for a survey)
- Dual Expected Utility (Yaari 1987 Ecta): tractable special case; recent applications to auctions and finance (Gershkov, Moldovanu, Strack, & Zhang 2022)

Cognitive Perception of Risk

- Choice under risk and computational complexity (Oprea, 2024 AER)
- Uncertainty regarding valuation
- Models of cognitive imprecision of risk (e.g., Netzer, Robson, Steiner, & Kocourek 2024, JEEA; Khaw, Li, & Woodford 2021 RES); existing applications to finance and macro

Ordered Reference Dependent Choice (Lim, 2021 WP)

- Way in which alternatives are compared depend on set of alternatives, e.g., existence of sure things, 'riskiness' of riskiest alternative, etc.

More Issues with Expected Utility

Should we just throw away EU?

- EU has **normative** appeal and people *should* behave according to its principles.
- EU is still a *useful* model for choice under risk
  - Understanding better when it holds and when it fails is illuminating