7. Stochastic Orders

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Overview

So far: risk attitudes, i.e., patterns of individual behaviour in choices involving risk

This lecture: know how to rank lotteries/distributions in unambiguous manner among groups of individuals.

1. Rank distributions F and G s.t. *every* EU maximiser (with monotone u) would agree. (e.g., everyone would agree £2 for sure is better than £1 for sure)
2. Rank distributions according to 'riskiness', i.e., s.t. *every* risk-averse EU maximiser would agree.
   (Is 2 stronger or weaker than 1?)

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2. Setup
3. First-Order Stochastic Dominance
4. Monotone Likelihood Ratio Order
5. Second-Order Stochastic Dominance
6. Background Risks
7. More

Overview

Setup

- **Outcome space:** $X \subseteq \mathbb{R}$

  $x \in X$: DM's final wealth.

- **Cumulative Probability Distributions Function** F

  $F : \mathbb{R} \to [0, 1]$ s.t. F is nondecreasing, right-continuous, $\lim_{x\to-\infty} F(x) = 0$, and $\lim_{x\to\infty} F(x) = 1$ with support on X, i.e. $\mathbb{P}_F(X) = \int_X dF(x) = 1$.

  Expectation Operator: $\mathbb{E}_F[\cdot]$

- $\mathcal{F}$: set of all cumulative probability functions on X

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First-Order Stochastic Dominance

Ranking of distributions s.t. *every* EU maximiser agrees:

Definition

A distribution F **first-order stochastically dominates** (FOSD) a distribution G, denoted by $F \geq_{FOSD} G$ iff, for all nondecreasing functions $u : X \to \mathbb{R}$, $\mathbb{E}_F[u] \geq \mathbb{E}_G[u]$.

Theorem

$\forall F, G \in \mathcal{F}, F \geq_{FOSD} G \iff \forall x \in X, F(x) \leq G(x).$

Remarkably simple characterisation of such strong property!

First-Order Stochastic Dominance

Theorem

$\forall F, G \in \mathcal{F}, F \geq_{FOSD} G \iff \forall x \in X, F(x) \leq G(x).$

Proof

$\implies$:

- $\forall a \in \mathbb{R}$, define $u_a(x) := \mathbf{1}_{\{x \geq a\}}$; $\mathbf{1}_A = 1$ if A is true, and 0 if ow.
- u_a nondecreasing $\forall a \in \mathbb{R}$.

$$\begin{aligned} F \geq_{FOSD} G &\implies \mathbb{E}_F[u_a] \geq \mathbb{E}_G[u_a] \ \forall a \in \mathbb{R} \\ &\iff \int_X u_a(x) dF(x) \geq \int_X u_a(x) dG(x) \ \forall a \in \mathbb{R} \\ &\iff \int_{x \geq a} 1 dF(x) \geq \int_{x \geq a} 1 dG(x) \ \forall a \in \mathbb{R} \\ &\iff 1 - F(a) \geq 1 - G(a) \ \forall a \in \mathbb{R} \\ &\iff F(a) \leq G(a) \ \forall a \in \mathbb{R}. \end{aligned}$$

First-Order Stochastic Dominance

Theorem

$\forall F, G \in \mathcal{F}, F \geq_{FOSD} G \iff \forall x \in X, F(x) \leq G(x).$

Proof

$\Longleftarrow$: A small detour

We'll use a result in statistics called the inverse transform method.

Inverse Transform Sampling

Definition

$\forall F \in \mathcal{F}$, the **generalised inverse** (or **quantile function**) is given by $Q_F(\tau) := \min\{x \in \mathbb{R} \mid F(x) \geq \tau\}, \forall \tau \in (0,1)$.

Proposition (Inverse Transform Sampling)

Let $F \in \mathcal{F}$ and $X \sim F$. Then, $X \overset{d}{=} Q_F(U)$, where $U \sim \text{Unif}(0,1)$.

Simulating uniform rv is convenient and computationally efficient $\implies$ computationally efficient way of simulating *any* rv!

Inverse Transform Sampling

Definition

$Q_F(\tau) := \min\{x \in \mathbb{R} \mid F(x) \geq \tau\}, \forall \tau \in (0,1)$.

Proposition (Inverse Transform Sampling)

Let $F \in \mathcal{F}$ and $X \sim F$. Then, $X \overset{d}{=} Q_F(U)$, where $U \sim \text{Unif}(0,1)$.

Proof

WTS $\mathbb{P}(Q_F(U) \leq x) = F(x)$.

(1) Q_F is nondecreasing:

F is nondecreasing $\implies \forall \tau' \geq \tau, \{x \in \mathbb{R} \mid F(x) \geq \tau'\} \subseteq \{x \in \mathbb{R} \mid F(x) \geq \tau\} \implies Q_F(\tau) \leq Q_F(\tau')$.

(2) $Q_F(F(x)) \leq x \because \forall Q_F(F(x)) = \min\{y : F(y) = F(x)\}$ and $x \in \{y : F(y) = F(x)\}$.

(3) Take $\tau \in (0,1), x \in \mathbb{R} : \tau < F(x)$. Then,

$$\tau < F(x) \implies Q_F(\tau) = \min\{y \in \mathbb{R} \mid F(y) \geq \tau\} \leq \min\{y \in \mathbb{R} \mid F(y) \geq F(x)\} = Q_F(F(x)) \leq x.$$

Inverse Transform Sampling

Definition

$Q_F(\tau) := \min\{x \in \mathbb{R} \mid F(x) \geq \tau\}, \forall \tau \in (0, 1)$.

Proposition (Inverse Transform Sampling)

Let $F \in \mathcal{F}$ and $X \sim F$. Then, $X \stackrel{d}{=} Q_F(U)$, where $U \sim \text{Unif}(0, 1)$.

Proof

WTS $\mathbb{P}(Q_F(U) \leq x) = F(x)$.

(1) Q_F is nondecreasing. (2) $Q_F(F(x)) \leq x$. (3) $\tau < F(x) \implies Q_F(\tau) \leq Q_F(F(x)) \leq x$.

(4) As $Q_F(\tau) \leq x \implies \tau \leq F(x)$ (Q_F nondecreasing), then

$$\{U < F(x)\} \subseteq \{Q_F(U) \leq x\} \subseteq \{U \leq F(x)\}$$

$$\iff \mathbb{P}(U < F(x)) \leq \mathbb{P}(Q_F(U) \leq x) \leq \mathbb{P}(U \leq F(x))$$

$$\iff F(x) \leq \mathbb{P}(Q_F(U) \leq x) \leq F(x).$$

□

First-Order Stochastic Dominance

Theorem

$\forall F, G \in \mathcal{F}, F \geq_{FOSD} G \iff \forall x \in X, F(x) \leq G(x).$

Proof

$\Longleftarrow$: Back to characterising FOSD. Fix u, define quantile functions Q_F and Q_G.

$$\begin{aligned} F(x) \leq G(x), \forall x \in X &\implies (F(x) \geq \tau \implies G(x) \geq \tau) \\ &\implies \{x \in X \mid F(x) \geq \tau\} \subseteq \{x \in X \mid G(x) \geq \tau\} \\ &\implies Q_F(\tau) \geq Q_G(\tau). \end{aligned}$$

$$\begin{aligned} F(x) \leq G(x), \forall x \in X &\implies Q_F(z) \geq Q_G(z), \forall z \in (0,1) \\ &\implies u(Q_F(z)) \geq u(Q_G(z)), \forall z \in (0,1) \qquad \text{as } u \text{ nondec} \\ &\implies \int_{[0,1]} u(Q_F(z))dz \geq \int_{[0,1]} u(Q_G(z))dz \\ &\iff \int_X u(x)dF(x) \geq \int_X u(x)dG(x) \qquad \text{inverse transform sampling} \\ &\iff \mathbb{E}_F[u] \geq \mathbb{E}_G[u]. \end{aligned}$$

□

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4. Monotone Likelihood Ratio Order

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Monotone Likelihood Ratio Order

Restrict attention to distrib. admitting either (i) density f or (ii) that have discrete support with pmf f

Definition

Let $F, G \in \mathcal{F}$ s.t. (i) either both admit a density, or (ii) both have discrete support. F **monotone likelihood ratio** dominates G ($F \geq_{MLR} G$) iff $f(x)/g(x)$ is nondecreasing in x.

Proposition

Let $F, G \in \mathcal{F}$ s.t. (i) either both admit a density, or (ii) both have discrete support. $F \geq_{MLR} G \implies F \geq_{FOSD} G$.

Proof

(1) $f(x)g(y) \geq f(y)g(x) \, \forall x \geq y \implies$ (a) $f(x)G(x) - F(x)g(x) \geq 0$ and (b) $(1 - F(x))g(x) - f(x)(1 - G(x)) \geq 0 \, \forall x$.

(2) Note (a) $f(x)G(x) - F(x)g(x) \geq 0 \implies \frac{f(x)}{g(x)} \geq \frac{F(x)}{G(x)}$ and

(b) $(1 - F(x))g(x) - f(x)(1 - G(x)) \geq 0 \implies \frac{f(x)}{g(x)} \leq \frac{1-F(x)}{1-G(x)}$.

(a) and (b) $\implies \frac{1-F(x)}{1-G(x)} \geq \frac{F(x)}{G(x)} \iff (1 - F(x))G(x) \geq F(x)(1 - G(x)) \iff G(x) \geq F(x) \, \forall x$.

Monotone Likelihood Ratio Order

Proposition

Let $F, G \in \mathcal{F}$ s.t. (i) either both admit a density, or (ii) both have discrete support. $F \geq_{MLR} G \implies F \geq_{FOSD} G$.

MLR is in a sense a minimal condition so that FOSD is preserved under Bayesian updating — very convenient property (see exercise)

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   - Stochastic Orders in $\mathbb{R}^n$
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Second-Order Stochastic Dominance

Definition

For $F, G \in \mathcal{F}$, F **second-order stochastically dominates** (SOSD) G ($F \geq_{SOSD} G$) iff $\mathbb{E}_F[u] - \mathbb{E}_G[u] \geq \mathbf{0}$ for all nondecreasing, concave functions $u : \mathbb{R} \to \mathbb{R}$, such that $\mathbb{E}_F[u] - \mathbb{E}_G[u]$ is well-defined and $\int_{-\infty}^{0} u(x)dF(x), \int_{-\infty}^{0} u(x)dG(x) > -\infty$.

Restricting F, G to have bounded support, then $\int_{-\infty}^{0} u(x)dF(x), \int_{-\infty}^{0} u(x)dG(x) > -\infty$

Second-Order Stochastic Dominance

Definition

For $F, G \in \mathcal{F}$, F **second-order stochastically dominates** (SOSD) G ($F \geq_{SOSD} G$) $\iff$ $\mathbb{E}_F[u] - \mathbb{E}_G[u] \geq 0$ for all nondecreasing, concave functions $u : \mathbb{R} \to \mathbb{R}$, such that $\mathbb{E}_F[u] - \mathbb{E}_G[u]$ is well-defined and $\int_{-\infty}^{0} u(x)dF(x), \int_{-\infty}^{0} u(x)dG(x) > -\infty$.

- WT find unambiguous ranking for risk averse DMs
  - Better understand riskiness
  - Separate individuals according to attitudes toward risk
- How does $\geq_{SOSD}$ compare with $\geq_{FOSD}$? Which one allows for a finer comparison? Which one is stronger?

Second-Order Stochastic Dominance

Theorem

$\forall F, G \in \mathcal{F}', F \geq_{SOSD} G \iff \forall x \in X, \int_{-\infty}^{x} F(s)ds \leq \int_{-\infty}^{x} G(s)ds.$

Result has had troubled history; first version Hadar & Russell (1969 AER) and general version Tesfatsion (1976 RES).

We'll prove the result for the subset of distributions with bounded support $\mathcal{F}'$.

Second-Order Stochastic Dominance

Theorem

$\forall F, G \in \mathcal{F}', F \geq_{SOSD} G \iff \forall x \in X, \int_{-\infty}^{x} F(s)ds \leq \int_{-\infty}^{x} G(s)ds.$

Proof

Preliminaries: integration by parts: $\int_a^b u(x)dF(x) = F(b)u(b) - F(a)u(a) - \int_a^b F(x)du(x)$.
Bounded support $\implies$ choose $\overline{x}, \underline{x} : F(\underline{x}) = G(\underline{x}) = \mathbf{0}$ and $F(\overline{x}) = G(\overline{x}) = \mathbf{1}$
u defined on $(\underline{x} - \varepsilon, \overline{x} + \varepsilon)$.

Second-Order Stochastic Dominance

Theorem

$\forall F, G \in \mathcal{F}', F \geq_{SOSD} G \iff \forall x \in X, \int_{-\infty}^{x} F(s)ds \leq \int_{-\infty}^{x} G(s)ds.$

Proof

$\implies$:

- For $a \in \mathbb{R}$, define $u_a(x) := \mathbf{1}_{x \leq a}(x - a)$; nondecreasing and concave
- Integration by parts:

$$\int_{\underline{x}}^{a} u_a(x)dF(x) - \int_{\underline{x}}^{a} u_a(x)dG(x)$$

$$= (F(a) - G(a))(a - a) - (F(\underline{x}) - G(\underline{x}))u_a(\underline{x}) + \int_{\underline{x}}^{a} (G(x) - F(x))dx$$

$$= \int_{\underline{x}}^{a} (G(x) - F(x)).$$

Second-Order Stochastic Dominance

Theorem

$\forall F, G \in \mathcal{F}', F \geq_{SOSD} G \iff \forall x \in X, \int_{-\infty}^{x} F(s)ds \leq \int_{-\infty}^{x} G(s)ds.$

Proof

$\implies$:

- For $a \in \mathbb{R}$, define $u_a(x) := \mathbf{1}_{x \leq a}(x - a)$; nondecreasing and concave
- Integration by parts: $\int_{\underline{X}}^{a} u_a(x)dF(x) - \int_{\underline{X}}^{a} u_a(x)dG(x) = \int_{\underline{X}}^{a}(G(x) - F(x))$.

$$\begin{aligned}
\mathbb{E}_F[u_a] - \mathbb{E}_G[u_a] \geq \mathbf{0}, \quad \forall a &\iff \int_{x \leq a} u_a(x)dF(x) \geq \int_{x \leq a} u_a(x)dG(x), \quad \forall a \\
&\iff \int_{x \leq a} u_a(x)dF(x) - \int_{x \leq a} u_a(x)dG(x) \geq \mathbf{0}, \quad \forall a \\
&\iff \int_{x \leq a} (G(x) - F(x))dx \geq \mathbf{0}, \quad \forall a \\
&\iff \int_{\underline{X}}^{a} F(x)dx \leq \int_{\underline{X}}^{a} G(x)dx, \quad \forall a.
\end{aligned}$$

Second-Order Stochastic Dominance

Theorem

$\forall F, G \in \mathcal{F}', F \geq_{SOSD} G \iff \forall x \in X, \int_{-\infty}^{x} F(s)ds \leq \int_{-\infty}^{x} G(s)ds.$

Proof Sketch

$\Longleftarrow$: Idea of the proof:

(i) Fix u and do a nice linear interpolation u^n of u over an n-evenly-spaced-point grid on $[\underline{x}, \overline{x}]$.

(ii) Show that, for any n, we can express u^n as a finite sum of positive affine transformations of functions in the family u_a.

(iii) Show that as $n \uparrow \infty$, u^n converges *uniformly* to u.

(iv) Use (ii) to show that $\int_{-\infty}^{x} F(s)ds \leq \int_{-\infty}^{x} G(s)ds \forall x \implies \sum_{i=1}^{K_n} \mathbb{E}_F[u_{x_i^n}] - \mathbb{E}_G[u_{x_i^n}] \equiv \mathbb{E}_F[u^n] - \mathbb{E}_G[u^n] \geq 0 \forall n$.

(v) Use (iii) and (iv) to show that $0 \leq \mathbb{E}_F[u^n] - \mathbb{E}_G[u^n] \to \mathbb{E}_F[u] - \mathbb{E}_G[u] \geq 0$.

(Filling in the blanks left as an exercise.)

Mean-Preserving Spread

Definition

For $F, G \in \mathcal{F}$, G is a **mean-preserving spread of** (MPS) F ($G \geq_{MPS} F$) iff $\exists$ random variables X, Y, and ε, such that $Y \overset{d}{=} X + \varepsilon$, $X \sim F$, $Y \sim G$, and $\mathbb{E}[\varepsilon \mid X] = \mathbf{0}$.

Properties of MPS

(i) $G \geq_{MPS} F \implies F \geq_{SOSD} G$, but the converse is not true in general.

(ii) $F \geq_{SOSD} G \implies \mathbb{E}_F[x] \geq \mathbb{E}_G[x]$.

(iii) $G \geq_{MPS} F \implies \mathbb{E}_F[x] = \mathbb{E}_G[x]$ and $\mathbb{V}_F[x] \leq \mathbb{V}_G[x]$. (Prove it)

(iv) $F \geq_{FOSD} G \implies F \geq_{SOSD} G$, but the converse is not true in general.

(v) $\geq_{SOSD}$ and $\geq_{MPS}$ are partial orders.

Second-Order Stochastic Dominance in $\mathbb{R}^n$

Results extend to more general spaces.

Definition

For $F, G \in \Delta(\mathbb{R}^n)$. F is a **second-order stochastically dominates** (FOSD) G ($F \geq_{SOSD} G$) iff $\mathbb{E}_F[u] \geq \mathbb{E}_G[u]$ for all nondecreasing concave $u : \mathbb{R}^n \to \mathbb{R}$, whenever both expectations exist.

Theorem (Strassen 1965, Theorem 2.6.8)

Let F and G be distributions on $\mathbb{R}^n$ with bounded support. $F \geq_{SOSD} G$ if and only if $\exists X \sim F$ and $Y \sim G$ such that $X \geq \mathbb{E}[Y \mid X]$ a.s.

Result provides a way to define a joint distribution $H(x, y)$ such that the marginals over x and y equal F and G and $\int_{\mathbb{R}^n} yH(x, y)dy = \mathbb{E}[Y|X = x] \leq x$.

Mean-Preserving Spreads in $\mathbb{R}^n$

Corollary

Let F and G be distributions on $\mathbb{R}^n$ with bounded support. G is a mean-preserving spread of F if and only if $F \geq_{SOSD} G$ and $\mathbb{E}_F[x] = \mathbb{E}_G[x]$.

Why do we care? +Information $\implies$ MPS of beliefs!

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Background Risks

Often simplify comparison of lotteries $X \sim F$ and $Y \sim G$.

In reality: background risks $\boldsymbol{\varepsilon}$; right comparison is $X + \boldsymbol{\varepsilon}$ vs $Y + \boldsymbol{\varepsilon}$

When background risks are significant, it may overwhelm limited risk in X in Y

Pomatto, Strack, & Tamuz (2020 JPE): study connection between (independent) background risks and stochastic orders

Theorem

Let X and Y be random variables with finite variance.

(i) If $\mathbb{E}[X] > \mathbb{E}[Y]$, then $\exists$ indep. random variable $\boldsymbol{\varepsilon} : X + \boldsymbol{\varepsilon} \geq_{FOSD} Y + \boldsymbol{\varepsilon}$.

(ii) If $\mathbb{E}[X] = \mathbb{E}[Y]$ and $\mathbb{V}[X] < \mathbb{V}[Y]$, then $\exists$ indep. random variable $\boldsymbol{\varepsilon} : X + \boldsymbol{\varepsilon} \geq_{SOSD} Y + \boldsymbol{\varepsilon}$.

Rationalises approximation of looking at expectation and variance in assessing assets when facing significant background noise.

Furthermore: if $\mathbb{E}[X] > \mathbb{E}[Y]$ and want everyone to prefer X to Y, can throw in some well calibrated background noise.

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More on Stochastic Orders

- Meyer and Strulovici (2012): ordering interdependence
  - useful for finance (valuing portfolios), empirical work (inputing data), measuring alignment of preferences in decision-making in groups (e.g. voting), etc.
- Kleiner, Moldovanu, and Strack (2021 Ecta) derive properties related to MPS and leverage these to study auctions, delegation, and decision-making under uncertainty (among others).
- MCS with stochastic orders: distributional comparative statics in macro models, in games, in information, etc. (Jensen 2018 RES)
- Reference textbooks: Muller and Stoyan (2002), Shaked and Shanthikumar (2007).