8a. Uncertainty

Duarte Gonçalves

University College London

MRes Microconomics

Overview

How likely is it that it is going to rain today?

Different websites show different probabilities, but do we know the true, objective probability that it rains today?
(Unclear if there is such a thing)

Even from frequentist perspective makes little sense to talk about *objective* probabilities of singular, unrepeatable events.

Modern concepts of prob. enabled by *subjective probability*.

Today: modelling choice under *uncertainty* and getting subjective beliefs *from* choice.

Extraordinarily important for theory and — especially — for applications.

Overview

1. Uncertainty
2. Subjective Expected Utility
3. Savage's Framework
4. Uncertainty Aversion
5. More

Overview

1. Uncertainty

2. Subjective Expected Utility
   - Anscombe-Aummann Framework
   - State-Dependent SEU
   - State-Independent SEU

3. Savage's Framework

4. Uncertainty Aversion

5. More

Anscombe-Aummann Framework

Main ingredients

- Ω: set of states of the world, finite;
- X: set of consequences or outcomes, finite;
- $f : \Omega \to \Delta(X)$: an act;
- $\mathcal{F} := \Delta(X)^{\Omega}$: set of acts;
- $\succsim \subseteq \mathcal{F}^2$: preference relation.

Two Sources of Uncertainty

(i) subjective uncertainty (horse race): which state $\omega \in \Omega$ will be realised

(ii) objective uncertainty/risk (roulette wheel): which consequence $x \in X$ will be realised in a lottery.

Can be seen as compound lottery: a potentially different objective lottery is triggered by each (uncertain) state of the world

Anscombe-Aummann Framework

Goal: characterise exact properties of $\succsim$ that enable SEU representation

Definition

$\succsim$ admits a SEU representation if $\exists u : X$ and $\mu \in \Delta(\Omega)$ s.t. $\forall f, g \in \mathcal{F}$, $f \succsim g \iff \mathbb{E}_\mu[\mathbb{E}_{f(\omega)}[u]] \geq \mathbb{E}_\mu[\mathbb{E}_{g(\omega)}[u]]$.

- Recover (i) Bernoulli utility function on consequences, $u : X \to \mathbb{R}$, *and* (ii) probability measure $\mu \in \Delta(\Omega)$.
- For given state ω, $f(\omega)$ is objective prob. distrib. in $\Delta(X) \implies \mathbb{E}_{f(\omega)}[u]$ is vNM EU
- μ represents (as if) DM's belief over states; take expectations of vNM EU wrt μ to get *subjective* expected utility
- Importantly: want to pin down DM's beliefs from preferences

Anscombe-Aummann Framework

Main ingredients

- Ω: set of states of the world, finite;
- X: set of consequences or outcomes, finite;
- $f : \Omega \to \Delta(X)$: an act;
- $\mathcal{F} := \Delta(X)^{\Omega}$: set of acts;
- $\succsim \subseteq \mathcal{F}^2$: preference relation.

More Definitions

Mixture $\alpha f + (1 - \alpha)g \in \mathcal{F}$, $f, g \in \mathcal{F}$ and $\alpha \in [0, 1]$

$(\alpha f + (1 - \alpha)g)(\omega) = \alpha f(\omega) + (1 - \alpha)g(\omega)$

$\forall p \in \Delta(X)$, denote **constant act** $\tilde{p} \in \mathcal{F}$ s.t. $\tilde{p}(\omega) = p \in \Delta(X)$ for every $\omega \in \Omega$.

State-Dependent SEU

Definition

- $\succsim$ sat. **continuity** iff $\{f_n, g_n\}_n \subset \mathcal{F} : f_n \succsim g_n \forall n$ and $(f_n, g_n) \to (f, g), \implies f \succsim g$.
- $\succsim$ sat. **independence** iff $\forall f, g, h \in \mathcal{F}$ and $\forall \alpha \in (0, 1], f \succsim g \iff \alpha f + (1-\alpha)h \succsim \alpha g + (1-\alpha)h$.

Definition

$\succsim$ admits a state-dependent SEU representation if $\exists u : X \times \Omega$ and $\mu \in \Delta(\Omega)$ s.t. $\forall f, g \in \mathcal{F}$, $f \succsim g \iff \mathbb{E}_\mu[\mathbb{E}_{f(\omega)}[u]] \geq \mathbb{E}_\mu[\mathbb{E}_{g(\omega)}[u]]$.

State-Dependent SEU

Definition

$\succsim$ admits a state-dependent SEU representation if $\exists u : X \times \Omega$ and $\mu \in \Delta(\Omega)$ s.t. $\forall f, g \in \mathcal{F}$, $f \succsim g \iff \mathbb{E}_\mu[\mathbb{E}_{f(\omega)}[u]] \geq \mathbb{E}_\mu[\mathbb{E}_{g(\omega)}[u]]$.

An intermediate result:

Theorem

A pref. rel. $\succsim$ on $\mathcal{F}$ sat. continuity and independence $\iff$ $\succsim$ admits a state-dependent SEU representation.

A problem: we have $u(x, \omega)$, not $u(x)$.

Unable to separate preference over consequences and beliefs about states.

$u(x, \omega)$ state-*dependent* utility, capturing both preferences over consequences and beliefs about states $\implies$ uniform belief/prior μ, void of any empirical content.

Anscombe-Aummann Framework

Theorem

A pref. rel. $\succsim$ on $\mathcal{F}$ sat. continuity and independence $\iff$ $\succsim$ admits a state-dependent SEU representation.

Focus on $\implies$. Before that:

More Notation/Terminology

- $E \subseteq \Omega$: an event
- fEg: a 'conditional act,' where for acts f, g and event E, $fEg \in \mathcal{F}$ is such that $(fEg)(\omega) = f(\omega)$ is $\omega \in E$ and $(fEg)(\omega) = g(\omega)$ if otherwise;
- Null event E: event s.t. $\forall f, g, h \in \mathcal{F} : f \succ g, fEh \sim gEh$ (why is it called null?);
- $\tilde{x}$: a constant act, $\tilde{x}(\omega) = x, \forall \omega \in \Omega$.

Lemma

Let $V : \mathcal{F} \to \mathbb{R}$ be affine and continuous. Then, $\forall \omega \in \Omega$, $\exists$ affine and continuous function $V_\omega : \Delta(X) \to \mathbb{R}$ s.t. $V(f) = \sum_\omega V_\omega(f(\omega)), \forall f \in \mathcal{F}$.

State-Dependent SEU

Lemma

Let $V : \mathcal{F} \to \mathbb{R}$ be affine and continuous. Then, $\forall \omega \in \Omega$, $\exists$ affine and continuous function $V_\omega : \Delta(X) \to \mathbb{R}$ s.t. $V(f) = \sum_\omega V_\omega(f(\omega)), \forall f \in \mathcal{F}$.

Proof

Fix $f^* \in \mathcal{F}$.

- For any $f \in \mathcal{F}$, as V is affine:

$$\frac{1}{|\Omega|} f = \frac{1}{|\Omega|} f^* + \frac{1}{|\Omega|} \sum_\omega (f\{\omega\}f^* - f^*) \iff \frac{1}{|\Omega|} f + \left(1 - \frac{1}{|\Omega|}\right) f^* = \frac{1}{|\Omega|} \sum_\omega (f\{\omega\}f^*)$$

$$\iff V\left(\frac{1}{|\Omega|} f + \left(1 - \frac{1}{|\Omega|}\right) f^*\right) = V\left(\frac{1}{|\Omega|} \sum_\omega (f\{\omega\}f^*)\right)$$

$$\iff \frac{1}{|\Omega|} V(f) + \left(1 - \frac{1}{|\Omega|}\right) V(f^*) = \frac{1}{|\Omega|} \sum_\omega V\left((f\{\omega\}f^*)\right)$$

$$\iff V(f) = \sum_\omega \left[V\left((f\{\omega\}f^*)\right) - (|\Omega| - 1) V(f^*)\right],$$

State-Dependent SEU

Lemma

Let $V : \mathcal{F} \to \mathbb{R}$ be affine and continuous. Then, $\forall \omega \in \Omega$, $\exists$ affine and continuous function $V_\omega : \Delta(X) \to \mathbb{R}$ s.t. $V(f) = \sum_\omega V_\omega(f(\omega))$, $\forall f \in \mathcal{F}$.

Proof

Fix $f^* \in \mathcal{F}$.

- For any $f \in \mathcal{F}$, as V is affine:

$$V(f) = \sum_\omega \left[V\left((f\{\omega\}f^*) \right) - (|\Omega| - 1)\, V(f^*) \right],$$

- Define $V_\omega : \Delta(X) \to \mathbb{R}$ s.t. $V_\omega(f(\omega)) := V\left((f\{\omega\}f^*) \right) - (|\Omega| - 1)\, V(f^*)$.
- V is continuous and affine $\implies$ V_ω is continuous and affine. $\square$

State-Dependent SEU

Theorem

A pref. rel. $\succsim$ on $\mathcal{F}$ sat. continuity and independence $\iff$ $\succsim$ admits a state-dependent SEU representation.

Proof

- $\forall f \in \mathcal{F}$, let $\rho_f \in \Delta(X \times \Omega)$ be s.t. $\rho_f(\{x\} \times \{\omega\}) := \frac{1}{|\Omega|} f(\omega)(x)\ \forall (x, \omega)$
  Note: ρ_f is joint distribution over $X \times \Omega$ with unif. marginal over Ω.
- Define (i) $R := \{\rho_f \mid f \in \mathcal{F}\} \subseteq \Delta(X \times \Omega)$, set of all such joint distributions (convex, subset of $(|X \times \Omega| - 1)$ simplex).

  (ii) $\trianglerighteq \subseteq R^2$ s.t. $\rho_f \trianglerighteq \rho_g \iff f \succsim g$.

State-Dependent SEU

Theorem

A pref. rel. $\succsim$ on $\mathcal{F}$ sat. continuity and independence $\iff$ $\succsim$ admits a state-dependent SEU representation.

Proof

- $\forall f \in \mathcal{F}$, let $\rho_f \in \Delta(X \times \Omega)$ be s.t. $\rho_f(\{x\} \times \{\omega\}) := \frac{1}{|\Omega|} f(\omega)(x)\ \forall (x, \omega)$
- $\trianglerighteq \subseteq R^2$ s.t. $\rho_f \trianglerighteq \rho_g \iff f \succsim g$.
- $\succsim$ continuous $\implies$ $\trianglerighteq$ continuous (and therefore vNM continuity).
- $\succsim$ satisfies independence $\implies$ $\forall \alpha \in (0, 1]$ and $\forall f, g, h \in \mathcal{F}$,

  $\rho_f \succsim \rho_g \iff \rho_{\alpha f+(1-\alpha)h} \trianglerighteq \rho_{\alpha g+(1-\alpha)h}$, which implies:

  $$\begin{aligned}\rho_{\alpha f+(1-\alpha)h}(\{x\} \times \{\omega\}) &= (\alpha f + (1-\alpha)h)(\omega)(x) = \alpha f(\omega)(x) + (1-\alpha)h(\omega)(x) \\ &= \alpha \rho_f(\{x\} \times \{\omega\}) + (1-\alpha)\rho_h(\{x\} \times \{\omega\}),\end{aligned}$$

  $\implies$ $\trianglerighteq$ sat. independence.
- Adapted vNM EU representation theorem: $\exists\, v : X \times \Omega \to \mathbb{R}$ s.t.

  $\rho_f \trianglerighteq \rho_g \iff \mathbb{E}_{\rho_f}[v] \geq \mathbb{E}_{\rho_g}[v]$.

State-Dependent SEU

Theorem

A pref. rel. $\succsim$ on $\mathcal{F}$ sat. continuity and independence $\iff$ $\succsim$ admits a state-dependent SEU representation.

Proof

- $\forall f \in \mathcal{F}$, let $\rho_f \in \Delta(X \times \Omega)$ be s.t. $\rho_f(\{x\} \times \{\omega\}) := \frac{1}{|\Omega|} f(\omega)(x) \; \forall (x, \omega)$
- $\trianglerighteq \subseteq R^2$ s.t. $\rho_f \trianglerighteq \rho_g \iff f \succsim g$.
- $\exists \, v : X \times \Omega \to \mathbb{R}$ s.t. $\rho_f \trianglerighteq \rho_g \iff \mathbb{E}_{\rho_f}[v] \geq \mathbb{E}_{\rho_g}[v]$.
- Define $V : \mathcal{F} \to \mathbb{R}$ such that $V(f) := \mathbb{E}_{\rho_f}[v]$, affine and continuous.
- V represents $\succsim$: $f \succsim g \iff \rho_f \trianglerighteq \rho_g \iff V(f) \geq V(g)$.
- By Lemma: $\exists V_\omega : \Delta(X) \to \mathbb{R}$ affine and continuous s.t. $V(f) = \sum_\omega V_\omega(f(\omega))$.
- Define $u : X \times \Omega$ as $u(x, \omega) := V_\omega(\delta_x)|\Omega|$.
- For each ω, V_ω is affine, $V_\omega(p) = \sum_x p(x) \frac{1}{|\Omega|} u(x, \omega)$.
- $V(f) = \sum_\omega V_\omega(f(\omega)) = \sum_\omega \sum_x f(\omega)(x) \frac{1}{|\Omega|} u(x, \omega) = \mathbb{E}_\mu[\mathbb{E}_{f(\omega)}[u]]$ with unif. $\mu \in \Delta(\Omega)$. $\square$

State-Independent SEU

Definition

$\succsim$ sat. **separability** iff $\forall p, q \in \Delta(X), h \in \mathcal{F}, \omega, \omega' \in \Omega$: $\{\omega\}$ and $\{\omega'\}$ are non-null events, $\tilde{p}\{\omega\}h \succsim \tilde{q}\{\omega\}h \iff \tilde{p}\{\omega'\}h \succsim \tilde{q}\{\omega'\}h$.

Theorem

(1) A pref. rel. $\succsim$ on $\mathcal{F}$ sat. continuity, independence, and separability $\iff$ $\succsim$ admits a SEU representation.

(2) Moreover, u is unique up to positive affine transformations and, if $\exists f, g \in \mathcal{F} : f \succ g$, μ is unique.

State-Independent SEU

Theorem

(1) A pref. rel. $\succsim$ on $\mathcal{F}$ sat. continuity, independence, and separability $\iff$ $\succsim$ admits a SEU representation.

Proof

- Focus on $\implies$. Start with state-dependent SEU: $\exists u : X \times \Omega$ s.t. $f \succsim g \iff \sum_{\omega,x} f(\omega)(x)u(x,\omega) \geq \sum_{\omega,x} g(\omega)(x)u(x,\omega)$.
- Let $U : \Delta(X) \times \Omega \to \mathbb{R}$ be defined as $U(p,\omega) := \sum_{x \in X} p(x)u(x,\omega)$ for all $\omega \in \Omega$, $p \in \Delta(X)$.

State-Independent SEU

Theorem

(1) A pref. rel. $\succsim$ on $\mathcal{F}$ sat. continuity, independence, and separability $\iff$ $\succsim$ admits a SEU representation.

Proof

- Take $p, q \in \Delta(X)$ and non-null $\{\omega\}$, $\omega \in \Omega$ s.t. $U(p, \omega) \geq U(q, \omega)$.
- Separability $\implies$ $\forall$ non-null $\{\omega'\}$, $\omega' \in \Omega$, $h \in \mathcal{F}$,

$$U(p,\omega) \geq U(q,\omega) \iff U(p,\omega) + \sum_{\omega'' \in \Omega \setminus \{\omega\}} U(h(\omega''),\omega'') \geq U(q,\omega) + \sum_{\omega'' \in \Omega \setminus \{\omega\}} U(h(\omega''),\omega'')$$

$$\iff V(\tilde{p}\{\omega\}h) \geq V(\tilde{q}\{\omega\}h)$$

$$\iff \tilde{p}\{\omega\}h \succsim \tilde{q}\{\omega\}h \iff \tilde{p}\{\omega'\}h \succsim \tilde{q}\{\omega'\}h$$

$$\iff V(\tilde{p}\{\omega'\}h) \geq V(\tilde{q}\{\omega'\}h)$$

$$\iff U(p,\omega') + \sum_{\omega'' \in \Omega \setminus \{x\}} U(h(\omega''),\omega'') \geq U(q,\omega') + \sum_{\omega'' \in \Omega \setminus \{x\}} U(h(\omega''),\omega'')$$

$$\iff U(p,\omega') \geq U(q,\omega').$$

State-Independent SEU

Theorem

(1) A pref. rel. $\succsim$ on $\mathcal{F}$ sat. continuity, independence, and separability $\iff$ $\succsim$ admits a SEU representation.

Proof

- $U(p,\omega) \geq U(q,\omega) \iff U(p,\omega') \geq U(q,\omega')$,
  i.e., $\forall$ non-null ω, $U(\cdot,\omega)$ all EU represent same preferences over $\Delta(X)$.
- Fix $\omega^* \in \Omega^*$ set non-null states; Define $u(x) := u(x,\omega^*)$. (ignore null states; why?)
- $\forall$ non-null ω, $\exists \alpha_\omega > 0, \beta_\omega : u(\cdot,\omega) = \alpha_\omega u + \beta_\omega$.

$$f \succsim g \iff \sum_{\omega\in\Omega} U(f(\omega),x) = \sum_{\omega\in\Omega}\sum_{x\in X} f(\omega)(x)u(\omega,x) \geq \sum_{\omega\in\Omega}\sum_{x\in X} g(\omega)(x)u(\omega,x)$$

$$\iff \sum_{\omega\in\Omega^*}\sum_{x\in X} f(\omega)(x)\alpha_\omega u(x) + \beta_\omega \geq \sum_{\omega\in\Omega^*}\sum_{x\in X} g(\omega)(x)\alpha_\omega u(x) + \beta_\omega$$

$$\iff \sum_{\omega\in\Omega^*} \frac{\alpha_\omega}{\sum_{\omega'\in\Omega^*}\alpha_{\omega'}} \sum_{x\in X} f(\omega)(x)u(x) \geq \sum_{\omega\in\Omega^*} \frac{\alpha_\omega}{\sum_{\omega'\in\Omega^*}\alpha_{\omega'}} \sum_{x\in X} g(\omega)(x)u(x).$$

State-Independent SEU

Theorem

(1) A pref. rel. $\succsim$ on $\mathcal{F}$ sat. continuity, independence, and separability $\iff$ $\succsim$ admits a SEU representation.

Proof

- $U(p, \omega) \geq U(q, \omega) \iff U(p, \omega') \geq U(q, \omega')$,
  i.e., $\forall$ non-null ω, $U(\cdot, \omega)$ all EU represent same preferences over $\Delta(X)$.
- Fix $\omega^* \in \Omega^*$ set non-null states; Define $u(x) := u(x, \omega^*)$. (ignore null states; why?)
- $\forall$ non-null ω, $\exists \alpha_\omega > 0, \beta_\omega : u(\cdot, \omega) = \alpha_\omega u + \beta_\omega$.
- Define $\mu \in \Delta(\Omega) : \mu(\omega) = 1_{\omega \in \Omega^*} \frac{\alpha_\omega}{\sum_{\omega' \in \Omega^*} \alpha_{\omega'}}$

$$f \succsim g \iff \mathbb{E}_\mu[\mathbb{E}_{f(\omega)}[u]] \geq \mathbb{E}_\mu[\mathbb{E}_{g(\omega)}[u]].$$

State-Independent SEU

Theorem

(2) Moreover, u is unique up to positive affine transformations and, if $\exists f, g \in \mathcal{F} : f \succ g$, μ is unique.

Proof

- $u(x, \omega)$ is unique up to positive affine transformations $\implies$ $u(x)$ is too.
- Suppose $f \succ g$; WTS $!\mu$. Let $\nu \in \Delta(\Omega) : f \succsim g \iff \mathbb{E}_\nu[\mathbb{E}_{f(\omega)}[u]] \geq \mathbb{E}_\nu[\mathbb{E}_{g(\omega)}[u]]$.

$$f \succ g \implies u \text{ is non-constant} \quad (\text{ow } \mathbb{E}_\mu[\mathbb{E}_{f(\omega)}[u]] = \mathbb{E}_\mu[\mathbb{E}_{g(\omega)}[u]])$$
$$\implies \exists z, y \in X : u(z) > u(y).$$

- For $\omega' \in \Omega^*$, take (i) non-constant acts $\tilde{\delta}_z\{\omega'\}\tilde{\delta}_y$, and (ii) constant acts $h = \mu(\omega')\tilde{\delta}_z + (1 - \mu(\omega'))\tilde{\delta}_y$.

$$\mathbb{E}_\mu[\mathbb{E}_{(\tilde{\delta}_z\{\omega'\}\tilde{\delta}_y)(\omega)}[u]] = \mu(\omega')u(z) + (1 - \mu(\omega'))u(y) = \mathbb{E}_{\mu(\omega')\delta_z + (1-\mu(\omega'))\delta_y}[u] = \mathbb{E}_\mu[\mathbb{E}_{h(\omega)}[u]]$$

$$\iff \tilde{\delta}_z\{\omega'\}\tilde{\delta}_y \sim h \iff$$

$$\mathbb{E}_\nu[\mathbb{E}_{(\tilde{\delta}_z\{\omega'\}\tilde{\delta}_y)(\omega)}[u]] = \nu(\omega')u(z) + (1 - \nu(\omega'))u(y) = \mu(\omega')u(z) + (1 - \mu(\omega'))u(y) = E_\nu[\mathbb{E}_{h(\omega)}[u]]$$

$$\iff \mu(\omega') = \nu(\omega') \forall \omega' \in \Omega^*. \qquad \square$$

Subjective Expected Utility

Foundational result.

Unclear if state-dependence is feature of real-world or due to imprecise specification specification of consequences

$\mathcal{F}$	Ω	
	rain	no rain
taking umbrella	not wet but carrying umbrella	not wet, carrying umbrella
not taking an umbrella	wet, not carrying umbrella	not wet, not carrying umbrella

X ={having to carry an umbrella, not having to carry an umbrella}$\times${getting wet, not getting wet}

Monotonicity

Theorem

Pref. rel. $\succsim$ on $\mathcal{F}$ sat. **monotonicity** iff $\forall f, g \in \mathcal{F}, \tilde{f}(\omega) \succsim \tilde{g}(\omega)\ \forall \boldsymbol{\omega} \in \boldsymbol{\Omega}$ s.t. $\{\omega\}$ is non-null $\implies f \succsim g$.

Proposition

If $\succsim$ sat. independence and continuity, then separability and monotonicity are equivalent.

Overview

1. Uncertainty

2. Subjective Expected Utility

3. Savage’s Framework

4. Uncertainty Aversion

5. More

Savage's Framework

Savage 1954, *The Foundations of Statistics*.

Original formulation of SEU

- Ω: set of states of the world;
- X: set of consequences or outcomes;
- $f : \Omega \to X$: an act;
- $\mathcal{F} := X^{\Omega}$: set of acts;
- $\succsim \subseteq \mathcal{F}^2$: preference relation.

Crucial difference: acts map to consequences; no need for state-dependent lotteries on the set of consequences.

Savage: $\mathbb{E}_{\mu}[u(f(\omega))] = \int_{\Omega} u(f(\omega))d\mu(\omega)$;

Anscombe–Aumann: $\mathbb{E}_{\mu}[\mathbb{E}_{f(\omega)}[u]] = \int_{\Omega} \int_{X} f(\omega)(x)u(x)d\mu(\omega)$.

Some Definitions

- $E \subseteq \Omega$: an event;
- fEg: **conditional act**, s.t. for acts f, g and event E, $fEg \in \mathcal{F} : (fEg)(\omega) = f(\omega)$ if $\omega \in E$ and $(fEg)(\omega) = g(\omega)$ if ow;
- **Null event** E: event s.t. $\forall f, g, h \in \mathcal{F} :\succ g, fEh \sim gEh$;
- $\tilde{x}$: **constant act**, $\tilde{x}(\omega) = x, \forall \omega \in \Omega$.

Savage's Framework

Postulates

P1 (Ordering): $\succsim$ is complete and transitive (a preference relation).

P2 (Sure-Thing Principle): $\forall f, g, h, h'$, and event E, $fEh \succsim gEh \iff fEh' \succsim gEh'$.
(P2 gives a form of independence.)

P3 (Monotonicity): $\forall$ constant acts, $\tilde{x}$ and $\tilde{y}$, $\tilde{x} \succsim \tilde{y} \iff \tilde{x}Eh \succsim \tilde{y}Eh$ for $\forall h$, E non-null.
(P3 allows us to rank acts based on the ranking of constant acts.)

P4 (Weak Comparative Probability): $\forall$ events A, B and constant acts $\tilde{x}, \tilde{x}', \tilde{y}, \tilde{y}'$, s.t.
$\tilde{x} \succ \tilde{y}$ and $\tilde{x}' \succ \tilde{y}'$,
$\tilde{x}A\tilde{y} \succsim \tilde{x}B\tilde{y} \iff \tilde{x}'A\tilde{y}' \succsim \tilde{x}'B\tilde{y}'$.
(P4 is crucial to infer from preferences alone whether an event A is more likely than another event B. Clearly separates taste and belief.)

Savage's Framework

Postulates

P5 (Nondegeneracy): $\exists$ constant acts $\tilde{x}, \tilde{y} : \tilde{x} \succ \tilde{y}$.
(P5 just makes it a nontrivial preference relation.)

P6 (Small Event Continuity): $\forall f, g : f \succ g$ and all consequences degenerate acts $\tilde{x}, \tilde{y}$, $\exists$ finite partition $\{E_i\}_{i \in [n]}$ of Ω s.t. $\tilde{x} E_i f \succ g$ and $f \succ \tilde{y} E_i g \; \forall i \in [n]$.
(P6 is a form of Archimedean property. This indirectly imposes constraints on Ω.)

P7 (Uniform Monotonicity): $\forall$ event E and acts f, g, (i) if $fEh \succ \tilde{g}(\omega)Eh$ for any $\omega \in E$ — i.e., $\tilde{g}(\omega)$ is constant act equal to $g(\omega) = x$ in every state ω' — and any act h, then $fEh \succsim gEh$; (ii) if $\tilde{f}(\omega)Eh \succ gEh \; \forall \omega \in E$, then $fEh \succsim gEh$.

Savage's SEU

Theorem

$\succsim$ satisfies P1-P7 if and only if there exist

(i) unique, nonatomic, finitely additive $\mu \in \Delta(\Omega)$ s.t. $\mu(E) = 0 \iff E$ is null event;

(ii) $u : X \to \mathbb{R}$, bounded and unique up to positive affine transformations

s.t. $\forall f, g \in \mathcal{F}$,

$$f \succsim g \quad \iff \quad \mathbb{E}_\mu[u \circ f] := \int_\Omega u(f(\omega))d\mu(\omega) \geq \int_\Omega u(g(\omega))d\mu(\omega) = \mathbb{E}_\mu[u \circ g].$$

Overview

Ellsberg Paradox

A box contains 60 balls: 20 are black and the rest are either red or green. Which would you prefer:

A £20 if a black ball is drawn;

B £20 if a red ball is drawn; or

C £20 if a green ball is drawn.

Most people choose *A*.

Which would you prefer:

a £20 if a black or a green ball is drawn;

b £20 if a black or a red ball is drawn; or

c £20 if a red or a green ball is drawn.

Most people choose *c*.

This is incompatible with SEU. (Why?)

Ellsberg Paradox

Independence always gets the blame. Ways of dealing with it:

- Maxmin EU and ‘sets of priors’ (Gilboa & Schmeidler 1989 JMathEcon)
- Choquet EU and capacities instead of prob. measures (Schmeidler 1989 Ecta)
- Uncertainty Aversion (Klibanoff, Marinacci, & Mukerji 2005 Ecta; Denti & Pomatto 2022 Ecta)

Maxmin Expected Utility

	Ω	
	ω_1	ω_2
f	p	0
g	0	p

$p \in \Delta(X)$. Suppose $f \sim g$; this implies that $\mu(\omega_1) = \mu(\omega_2) = 1/2$.

It also implies DM is indifferent between f, g, and $\tilde{q}$, where $q := 1/2p + 1/2\delta_0$.

Indeed, it is not unreasonable to consider that $\tilde{q} \succ f \sim g$, as $\tilde{q}$ entails no uncertainty.

Definition

$\succsim \subseteq \mathcal{F}^2$ is **GS uncertainty averse** (neutral/seeking) if $\forall f, g \in \mathcal{F}, f \sim g \implies \frac{1}{2}f + \frac{1}{2}g \succsim f$ $(\sim/\precsim)$.

Definition

$\succsim \subseteq \mathcal{F}^2$ sat. **C-independence** if $\forall f, g \in \mathcal{F}$, $p \in \Delta(X)$, and $\alpha \in (0, 1]$, $f \succsim g \iff \alpha f + (1-\alpha)\tilde{p} \succsim \alpha g + (1-\alpha)\tilde{p}$.

Hedging is only valuable when it can eliminate uncertainty, which is not the case if it uses a constant act.

Maxmin Expected Utility

Theorem

Let $\succsim$ be preference relation on $\mathcal{F}$. $\succsim$ satisfies continuity, monotonicity, C-independence, and GS uncertainty aversion if and only if
$\exists u : X \to \mathbb{R}$ and convex and compact set $\mathcal{M} \subseteq \Delta(\Omega)$ s.t.

$$f \succsim g \quad \iff \quad \min_{\mu \in \mathcal{M}} \mathbb{E}_\mu[\mathbb{E}_f[u]] \geq \min_{\mu \in \mathcal{M}} \mathbb{E}_\mu[\mathbb{E}_g[u]].$$

DM has 'set of prob. meas.' $\mathcal{M} \subseteq \Delta(\Omega)$ that is *endogenous* to the representation.

Different $\succsim$ can induce representations with different $\mathcal{M}$.

Maxmin implicitly assumes extreme uncertainty aversion, behaving as if expecting worst to happen among all prob. distr. they entertain.

Beliefs over Unknown Probabilities

Can't we get something like standard risk aversion but for uncertainty instead of extreme uncertainty aversion? Yes, we can.

Klibanoff, Marinacci, & Mukerji (2005 Ecta) and Denti & Pomatto (2022 Ecta) provide two different axiomatisations of *smooth uncertainty aversion*

$$U(f) := \int_{\Delta(\Omega)} \phi\left(\int_{\Omega} u(f(\omega))d\mu(\omega)\right) d\pi(\mu)$$

Interpretation

- $f : \Omega \to X$ is Savage act
- $u : X \to \mathbb{R}$ vNM utility
- $\phi : \mathbb{R} \to \mathbb{R}$ a strictly increasing and continuous function
- $\mu \in \Delta(\Omega)$ is a prob. measure on state space
- $\pi \in \Delta(\Delta(\Omega))$ is DM's *prior*, capturing uncertainty about how state is actually distributed.

Smooth Uncertainty Aversion:

- curvature of u captures risk attitudes
- curvature of ϕ captures uncertainty attitudes (concave/linear/convex)

  Maxmin as limit of extreme risk aversion.

Overview

1. Uncertainty
2. Subjective Expected Utility
3. Savage's Framework
4. Uncertainty Aversion
5. More

More!

- **Massive literature on subjective uncertainty and alternatives** (both theoretical and experimental); see Machina & Siniscalchi (2014 Handbook of the Economics of Risk and Uncertainty Vol 1. Ch. 13)
- **Relation between risk and uncertainty attitudes** (Halevy 2007 Ecta; also Chapman et al. 2023 JPE Micro): attitudes to ambiguity and compound objective lotteries are tightly associated
- **Methods to elicit beliefs and patterns in belief updating:** important beyond just experimental and theory! E.g., development and education (Dizon-Ross 2019 AER), macro (Bordalo et al. 2020 AER), health (de Paula, Valente, & Miller 2022 WP), finance (Giglio et al. 2021 AER), political economy (Ortoleva Snowberg 2015 AER) Take + theory and experimental!

Where does this leave SEU?

SEU remains a benchmark framework: very appealing principles and well-known virtues and vices.

Behaviourally: neither comes for free and it's important to know this.

Model is approximation and, unless there is a crucial element missing, SEU are defaults so as to better understand differences in the model (i.e., what's the effect of the new ingredient on the soup's flavour overall).