Common Learning

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Topics in Economic Theory

3 people are prisoners on an island.

They cannot communicate with each other, cannot see their reflections, and all had green eyes.

The island is ruled by a despotic bear who imposed a peculiar rule:

a prisoner can ask to leave every night, but only prisoners with green eyes will be permitted to escape, whilst all others will be tossed in the volcano.

All the prisoners want to leave, but will never take action unless they are absolutely certain that they have green eyes.

You want to do something to help the prisoners.

The dictator allows you one thing only: to say a single sentence to the prisoners.

But there's a twist: you can't tell them anything that each didn't know.

What do you do?

‘There is at least one person with green eyes.’

Everyone knew that. On the first day, no one asks to leave. On the second day, neither. On the third day, everyone does.

If no one asks to leave on day 1, then it must be that everyone sees someone with green eyes
(otherwise they’d deduce they themselves have green eyes).

If no one asks to leave on day 2, then it must be that everyone sees two people with green eyes
(otherwise they’d deduce they themselves have green eyes).

On day 3, everyone is sure they have green eyes.

Common Knowledge

What happened? Saying something everyone knows makes it *commonly known*.

Who cares (other than for its own sake)?

- Highlights the role of public signals and announcements (monetary policy, auctions).
- Clarifies limits of coordination (distributed systems, protests, currency attacks).
- Provides epistemic foundations for solution concepts (backward induction, Bayesian Nash equilibrium).

This lecture: formalising knowledge and deriving implications.

Overview

1. Knowledge

2. Common Prior Assumption and Its Implications

3. Common Belief

4. Common Learning

5. Universal Type Space

6. Syntactic Knowledge

Overview

Knowledge

$\omega \in \Omega$: state of the world. Finite.

Events $E, F \subseteq \Omega$.

Knowledge Function: $k : \Omega \to 2^{\Omega}$

When true state is ω, $k(\omega)$ represents what DM knows.

Or, $k(\omega)$ are the states DM cannot distinguish from ω.

Knowledge

Example

$\Omega = \{\omega, \omega'\}$, $k(\omega) = \{\omega, \omega'\}$, $k(\omega') = \{\omega'\}$.

ω= Skipped my stop; ω= Didn't skip my stop.

If I didn't skip my stop, I know I didn't. By if I did, I don't know that I did.

Issue: if introspect, I realise that if I didn't skip my stop I would know it, so if I don't know I must've skipped it!

If I knew k, introspection would rule this out.

Knowledge

$\omega \in \Omega$: state of the world. Finite.

Events $E, F \subseteq \Omega$.

Knowledge Function: $k : \Omega \to 2^{\Omega}$

When true state is ω, $k(\omega)$ represents what DM knows.

Or, $k(\omega)$ are the states DM cannot distinguish from ω.

Partitional Knowledge function: (i) $\forall \omega \in \Omega$, $\omega \in k(\omega)$ and (ii) $\omega' \in k(\omega) \implies k(\omega) = k(\omega')$.

Knowledge Operator: $K : 2^{\Omega} \to 2^{\Omega}$ s.t. $K(E) := \{\omega \in \Omega \mid k(\omega) \subseteq E\}$.

Note: $K(E) = \cup_{k(\omega) \subseteq E} k(\omega)$.

Knowledge

Lemma

If k is partitional, then:

1. $K(\Omega) = \Omega$. (Axiom of awareness)
2. $K(E) \cap K(F) = K(E \cap F)$.
3. $K(E) \subseteq E$. (Axiom of knowledge)
4. $K(E) = K(K(E))$. (axiom of transparency)
5. $\Omega \setminus K(E) = K(\Omega \setminus K(E))$. (Axiom of wisdom)
6. $F \subseteq E \implies K(F) \subseteq K(E)$. (Monotonicity)
7. $K(E) = \cup_{\omega \in K(E)} k(\omega)$.

Proof

1. $\forall \omega, \omega \in k(\omega) \subseteq \Omega \implies K(\Omega) = \Omega$.
2. $\omega \in K(E \cap F) \iff \exists \omega' : \omega \in k(\omega') = k(\omega) \subseteq E \cap F \iff \omega \in K(E) \cap K(F)$.
3. $\omega \in K(E) \implies \omega \in k(\omega) \subseteq E$.

Knowledge

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7. $K(E) = \cup_{\omega \in K(E)} k(\omega)$.

Proof

4. $K(K(E)) \subseteq K(E)$ from 3. Moreover, $\omega \in K(K(E)) \implies k(\omega) \subseteq K(E) \subseteq E \implies \omega \in K(E) \implies K(K(E)) \subseteq K(E)$ (Using 3.).

Knowledge

Lemma

If k is partitional, then:

1. $K(\Omega) = \Omega$. (Axiom of awareness)
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7. $K(E) = \cup_{\omega \in K(E)} k(\omega)$.

Proof

5. Using 3., as $K(E) = K(K(E)) = \cup_{K(\{\omega\}) \subseteq K(E)} K(\{\omega\})$ $K(\Omega \setminus K(E)) \subseteq \Omega \setminus K(E)$ from 3. Moreover, as k partitional, $\omega \in \Omega \setminus K(E) \implies \omega \notin K(E) = K(K(E)) \implies \neg(k(\omega) \subseteq K(E)) \implies (\omega' \notin K(E) \forall \omega' \in k(\omega)) \implies k(\omega) \subseteq \Omega \setminus K(E) \implies \omega \in K(\Omega \setminus K(E))$.

Knowledge

Proposition

If k is partitional, then:

1. $K(\Omega) = \Omega$. (Axiom of awareness)
2. $K(E) \cap K(F) = K(E \cap F)$.
3. $K(E) \subseteq E$. (Axiom of knowledge)
4. $K(E) = K(K(E))$. (axiom of transparency)
5. $\Omega \setminus K(E) = K(\Omega \setminus K(E))$. (Axiom of wisdom)
6. $F \subseteq E \implies K(F) \subseteq K(E)$. (Monotonicity)
7. $K(E) = \cup_{\omega \in K(E)} k(\omega)$.

Proof

6. $F \subseteq E \implies K(F) = K(F \cap E) = K(F) \cap K(E) \subseteq K(E)$ (using 2).
7. $K(E) = \cup_{\omega \in K(E)} \{\omega\} \subseteq K(E) = \cup_{\omega \in K(E)} k(\omega)$. Moreover, $\forall \omega' \in \cup_{\omega \in K(E)} k(\omega)$, $\exists \omega'' \in K(E) : \omega' \in k(\omega'')$; and as $k(\omega') = k(\omega'') \subseteq E$, then $\omega' \in K(E)$.

Interactive Knowledge

Player $i \in \{1, \ldots, I\}$ with knowledge operator K_i.

Assume partitional k_i henceforth.

Definition

(i) There is **mutual knowledge** of $E \subseteq \Omega$ at ω if $\omega \in K^1(E) := \cap_i K_i(E)$.

(ii) Let $K^{n+1}(E) := K^1(K^n(E))$, for $n = 1, 2, \ldots$. There is **common knowledge** of $E \subseteq \Omega$ at ω if $\omega \in K^\infty(E) := \cap_n K^n(E)$.

Remark

If E is CK at ω, then $\forall F \supseteq E$, F is CK at ω.

Proof

By monotonicity of K_i, $\omega \in K^\infty(E) \subseteq K^\infty(F)$.

Can also consider CK for subset of players.

Properties of Mutual and Common Knowledge Operators

Proposition

If k_i is partitional $\forall i$, then, for any $n = 1, 2, ..., \infty$,

1. $K^n(\Omega) = \Omega$. (Axiom of awareness)
2. $K^n(E) \cap K^n(F) = K^n(E \cap F)$.
3. $K^n(E) \subseteq E$. (Axiom of knowledge)
4. $K^n(E) \supseteq K^n(K^n(E))$. (axiom of transparency)
5. $\Omega \setminus K^n(E) \supseteq K^n(\Omega \setminus K^n(E))$. (Axiom of wisdom)
6. $F \subseteq E \implies K^n(F) \subseteq K(E)$. (Monotonicity)

Properties of Mutual and Common Knowledge Operators

Proof

We prove for $n = 1$.

1. $\cap_i K_i(\Omega) = \cap_i \Omega = \Omega$.
2. $(\cap_i K_i(E)) \cap (\cap_i K_i(F)) = \cap_i (K_i(E) \cap K_i(F)) = \cap_i K_i(E \cap F)$.
3. $\cap_i K_i(E) \subseteq \cap_i E = E$.
4. Follows from 3.
5. Follows from 3.
6. $F \subseteq E \implies K_i(F) \subseteq K_i(E)\, \forall i \implies \cap_i K_i(F) \subseteq \cap_i K_i(E)$.

Iterate arguments to extend to $n > 1$.

Interactive Knowledge

Example

$\Omega = \{\omega_1, \omega_2, \omega_3\}$.

$k_1(\omega_1) = \{\omega_1\}, \quad k_1(\omega_2) = k_1(\omega_3) = \{\omega_2, \omega_3\}$.

$k_2(\omega_1) = k_2(\omega_2) = \{\omega_1, \omega_2\}, \quad k_2(\omega_3) = \{\omega_3\}$.

$E = \{\omega_2, \omega_3\} \implies K_1(E) = E, K_2(E) = \{\omega_3\} \implies K^1(E) = \{\omega_3\}$.

$K_1(K^1(E)) = \emptyset \implies K^n(E) = K^\infty(E) = \emptyset, \forall n \geq 2$.

Only Ω is CK.

How to get CK? By assumption or deriving CK from CK of something else.

Interactive Knowledge

Definition

Event E is evident if it is mutually known, $E \subseteq K^1(E)$.

If E happens, everyone knows E happens.

Remark

(i) E is evident $\implies E \subseteq K^1(E) \subseteq E \implies E = K^1(E)$.

(ii) E is evident $\iff k_i(\omega) \subseteq E, \forall \omega \in E$.

(iii) If E is evident, $E = K^\infty(E)$ and so E is CK at any $\omega \in E$.

Interactive Knowledge

Definition

Event E is evident if it is mutually known, $E \subseteq K^1(E)$.

Proposition (Monderer and Samet, 1989 GEB)

C is CK at ω if and only if there is an evident event E s.t. $\omega \in E$ and $E \subseteq K^1(C)$.

One could have just as well have written (...) "$\omega \in E$ and $E \subseteq C$"

Proof

If: $E \subseteq K^1(C) \implies \omega \in E = K^\infty(E) \subseteq K^\infty(C)$.

Only if: Let $E := K^\infty(C) := \cap_n K^n(C)$. Then, $K_i(E) = E \, \forall i \implies K^1(E) = E$, and E is evident. Moreover, by transparency, $E = K^\infty(C) \subseteq K^1(C)$.

Overview

Adding Beliefs to Partitional Model

Prior Belief P_i of player i over Ω. Assume full support, $P_i > \mathbf{0}$.

Posterior Belief $P_i(E|k_i(\omega)) = P_i(E \cap k_i(\omega))/P_i(k_i(\omega))$.

Common Prior $P \in \Delta(\Omega)$ if $P_i = P \; \forall i$.

Harsanyi Doctrine: We are born equal; we have different views about the world because we receive different information.

Implications of common prior assumption (CPA)?

Agreeing to disagree and no-trade theorem.

Agreeing to Disagree

Aumann (1976 AMS): Two individuals with a common prior belief, even if they had very different information (attending different school, different upbring, etc.) cannot agree to disagree.

I.e., if differences are due to information and posteriors are common knowledge, then there can't be disagreement.

Theorem

Let there be a common prior P. Suppose it is CK at ω^* that player 1's posterior beliefs on event E are m_1 whereas player 2's are m_2. Then, $m_1 = m_2$.

Proof

Consider $D_i := \{\omega | P(E \mid k_i(\omega)) = m_i\}$.

$D_1 \cap D_2$ CK at ω^*, $\exists$ evident event F s.t. $\omega^* \in F \subseteq D_1 \cap D_2$.

As F is evident, $F = K_i(F) = \{\omega \mid k_i(\omega) \subseteq F\}$. Hence, $F = \cup_{\omega \in F} k_i(\omega)$, where $\{k_i(\omega)\}_{\omega \in F}$ denotes a partition.

Since, for disjoint A, B, one has $P(E|A) = P(E|B) \implies P(E|A) = P(E|A \cup B)$, then $P(E|F) = P(E|k_i(\omega)) = m_i$.

Agreeing to Disagree

Theorem

Let there be a common prior P. It cannot be CK at some ω^* that player 1's posterior beliefs on event E is strictly greater than player 2's.

Proof

Suppose not: $\exists\omega^*$ at which $D := \{\omega | P(E \mid k_1(\omega)) > P(E \mid k_2(\omega))\}$ is CK.

$\exists$ evident event F s.t. $\omega^* \in F \subseteq D$.

$\implies \forall\omega \in F, P(\omega)P(E|k_1(\omega)) > P(\omega)P(E|k_2(\omega))$.

$\implies, P(F \cap E) = \sum_{\omega\in F} P(\omega)P(E|k_1(\omega)) > \sum_{\omega\in F} P(\omega)P(E|k_2(\omega)) = P(F \cap E)$, contradiction.

Agreeing to Disagree

It *can* be CK that two players have different beliefs at an event E.

Example

$\Omega = \{\omega_1, \omega_2\}$, P uniform common prior.

$\forall \omega$, $k_1(\omega) = \{\omega\}$ $k_2(\omega) = \Omega$.

$E = \Omega$.

"the two players have different posterior beliefs" at every state of the world, and this event is common knowledge: $K^{\infty}(\Omega) = \Omega$.

Agreeing to Disagree

Corollary

Let there be a common prior P and $X : \Omega \to \mathbb{R}$ a random variable. It cannot be CK at some ω^* that player 1's holds a higher expectation of X than player 2 does.

Proof

Suppose not: $\exists \omega^*$ at which $D := \{\omega | \mathbb{E}[X \mid k_1(\omega)] > \mathbb{E}[X \mid k_2(\omega)]\}$ is CK.

$\exists$ evident event F s.t. $\omega^* \in F \subseteq D$.

$\implies \forall \omega \in F, \frac{P(\omega)}{P(F)} \mathbb{E}[X|k_1(\omega)] > \frac{P(\omega)}{P(F)} \mathbb{E}[X|k_2(\omega)]$.

$\implies \mathbb{E}[X \mid F] = \sum_{\omega \in F} \frac{P(\omega)}{P(F)} \mathbb{E}[X|k_1(\omega)] > \sum_{\omega \in F} \frac{P(\omega)}{P(F)} \mathbb{E}[X|k_2(\omega)] = \mathbb{E}[X \mid F]$, contradiction.

No-Trade Theorem

Typical reason provided for trading: differences in information.

Milgrom and Stokey (1982 JET) show this is not exactly correct...

Definitions

Allocation: $a : \Omega \to A$, a contract that associates each state with an allocation or transfer to all agents.

Payoffs: Player i's state-dependent utility function $u_i(a(\omega), \omega)$.

Ex-ante Efficiency: b is ex-ante efficient if $\nexists a$ s.t. $\forall i$ $\mathbb{E}[u_i(a(\omega), \omega)] \geq \mathbb{E}[u_i(b(\omega), \omega)]$ with a strict inequality for some i.
i.e., $\sum_{\omega \in \Omega} P(\omega) u_i(a(\omega), \omega) \geq \sum_{\omega \in \Omega} P(\omega) u_i(b(\omega), \omega)$

No-Trade Theorem

Theorem (Milgrom and Stokey, 1982 JET)

Let there be a common prior P. Suppose b is ex-ante efficient. If cannot be common knowledge that there is some allocation a that is weakly preferred to b by all players and strictly by at least one.

Then players cannot trade away from b even if new information k_i arrives.

Agreeing to disagree results have brutal implications for trading:

- with CPA, once we get to an ex-ante efficient allocation, there is no scope for purely information-based trade.

No-Trade Theorem

Theorem (Milgrom and Stokey, 1982 JET)

Let there be a common prior P. Suppose b is ex-ante efficient. If cannot be common knowledge that there is some allocation a that is weakly preferred to b by all players and strictly by player 1.

Proof

Suppose $\exists a$ s.t. CK (with new information) that a is weakly preferred to b by everyone and strictly so by player 1.

Let F be evident event contained in

$\{\omega \in \Omega | \mathbb{E}[u_i(a(\omega), \omega) - u_i(b(\omega), \omega) \mid k_i(\omega)] \geq 0 \forall i \text{ and } \mathbb{E}[u_1(a(\omega), \omega) - u_1(b(\omega), \omega) \mid k_1(\omega)] > 0\}$.

Then, $\forall i$, $\mathbb{E}[u_i(a(\omega), \omega) - u_i(b(\omega), \omega) \mid F] \geq 0$, and $\mathbb{E}[u_1(a(\omega), \omega) - u_1(b(\omega), \omega) \mid F] > 0$.

Define contract c: $c(\omega) = a(\omega)$ if $\omega \in F$ and $c(\omega) = b(\omega)$ if otherwise. We get

$$\mathbb{E}[u_i(c(\omega), \omega) - u_i(b(\omega), \omega)] = P(F)\mathbb{E}[u_i(a(\omega), \omega) - u_i(b(\omega), \omega)]$$

which is ≥ 0 for all i and > 0 for $i = 1$.

Contradicts b being ex-ante efficient.

No-Trade Theorem

Typical reason provided for trading: differences in information.

Milgrom and Stokey (1982 JET) show this is not exactly correct...

With CPA, once we get to an ex-ante efficient allocation, there is no scope for purely information-based trade.

Note! Prices do adjust to new information.
More: under weak conditions
(essentially strict risk-aversion, smooth EU, ex-ante efficiency
— see Milgrom and Stokey, 1982 JET, Theorem 3)
change in relative prices reveals new info available to traders and is independent of endowments, utility functions, prior beliefs, and initial allocation.

Charaterising CPA

CPA has important consequences, not just theoretical convenience.

If observe beliefs of different players, can we say whether there is a common prior?

Two fundamental contributions:

- Samet (1998 GEB), Common Priors and Separation of Convex Sets; and
- Samet (1998 GEB), Iterated Expectations and Common Priors.

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Samet (1998 GEB), Common Priors and Separation of Convex Sets

Each agent's set of priors = convex hull of agent's types.

Proposition: A common prior exists if and only if the intersection of these convex sets is nonempty.

Proof idea: Generalisation of the separation theorem: multiple convex, closed subsets of the simplex intersect $\iff$ no linear functional can simultaneously separate them.

Interpretation: Absence of common prior $\iff$ existence of a bet that everyone expects to win. (No-trade-theorem-like converse)

Charaterising CPA

CPA has important consequences, not just theoretical convenience.

If observe beliefs of different players, can we say whether there is a common prior?

Samet (1998 GEB), Iterated Expectations and Common Priors

Can we test for a common prior using only present beliefs?

Key idea: (1) Start with any random variable X. (2) Compute iterated expectations: Eve's expectation of X, Adam's expectation of Eve's expectation, Eve's expectation of Adam's expectation, ... These sequences always converge.

Proposition: Common prior exists if and only if for every X, all iterated expectation sequences converge to the same limit. Common limit is expectation under the common prior.

Proof idea: Represent type functions as Markov matrices. Common prior = invariant probability measure for all players' matrices.

Charaterising CPA

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If observe beliefs of different players, can we say whether there is a common prior?

Two fundamental contributions:

- Samet (1998 GEB), Common Priors and Separation of Convex Sets; and
- Samet (1998 GEB), Iterated Expectations and Common Priors.

Also: Feinberg (2000 JET), Geanakoplos and Polemarchakis (1982 JET).

Related: Literature on merging of beliefs.

- Epistemic foundations for solution concepts (Aumann 1987 Ecta; Aumann and Brandenburger 1995 Ecta).

Overview

Common Belief

(Ω, Σ, μ) probability space, I finite set agents, k_i induces partition of Ω,
$\mathcal{F}_i := \sigma(\{k_i(\omega)\}_{\omega \in \Omega})$.

"i knows E at ω" = $\{k_i(\omega) \subseteq E\}$.

"i knows E" = $K_i(E) := \{\omega \in \Omega \mid k_i(\omega) \subseteq E\}$.

From "i knows E at ω" to "i believes E wp$\geq p$ at ω" $\equiv$ "i p-believes E at ω".

"i p-believes E at ω" = $\{\mu(E|k_i(\omega)) \geq p\}$.

"i p-believes E" = $B_i^p(E) = \{\omega \in \Omega \mid \mu(E|k_i(\omega)) \geq p\}$.

Properties of p-Belief

Proposition

For $E, F \in \Sigma$, $p \in [0,1]$, $i \in I$:

1. $\mu(E|B_i^p(E)) \geq p$.
2. $B_i^p(E) \in \mathcal{F}_i$.
3. If $E \in \mathcal{F}_i$, then $B_i^p(E) = E$.
4. $B_i^p(B_i^p(E)) = B_i^p(E)$.
5. $E \subseteq F \implies B_i^p(E) \subseteq B_i^p(F)$.
6. If (E^n) is decreasing sequence of events, then $B_i^p(\cap_n E^n) = \cap_n B_i^p(E^n)$.

1-belief = knowledge with finite models; not in continuous models:

e.g., 1-believe that uniform random draw from $[0,1]$ is irrational, but we don't know it.

Common p-Belief

Definition

(i) There is **mutual p-belief** of E at ω if $\omega \in B^{p,1}(E) := \cap_i B_i^p(E)$.

(ii) Let $B^{p,n+1}(E) := B^{p,1}(B^{p,n}(E))$, for $n = 1, 2, \ldots$. There is **common p-belief** of E at ω if $\omega \in B^{p,\infty}(E) := \cap_n B^{p,n}(E)$.

Common p-belief as 'almost CK'. Common p-belief vs CK:

C = currency attack starts at 9:00.

E=At 9:00, on a phone call, there is an announcement among traders that currency attack starts.

If everyone sees announcement, E is evident and C is CK.

But if there is a small chance not everyone is paying attention, E is not evident (may even not be p-evident for high p).

Common p-Belief

Definition

E is **evident p-believed** if it is mutually p-believed, $E \subseteq B^{p,1}(E)$.

Evident p-belief event: whenever E occurs, everyone assigns probability of at least p to its occurrence.

Proposition (Monderer and Samet, 1989 GEB)

C is common p-believed at ω if and only if there is an evident p-belief event E s.t. $\omega \in E$ and $E \subseteq B^{p,1}(C)$.

Proof

If: $E \subseteq B^{p,1}(C) \implies \omega \in E \subseteq B^{p,1}(E) \subseteq B^{p,\infty}(C)$.

Only if: Let $E := B^{p,\infty}(C)$. Then, $E = B^{p,1}(E) = B^{p,\infty}(C)$. By transparency, $E = B^{p,\infty}(C) \subseteq B^{p,1}(C)$.

Overview

Common Learning: Attacking a Currency

Coordinating on a Currency Attack

- Two traders coordinate on when to attack currency A or B.
- Every day, each trader simultaneously decides to attack currency A, B, or wait.
- They only stand to gain if both attack the weaker currency and at the same time.
- Every day, each receives a private signal about if it's best to attack A or B (the state).
- Signals are iid conditional on the state, but possibly correlated across traders.

Coordination Requirements

- Coordination requires traders to be sufficiently convinced if state is A or B.
- With fixed state, if traders will a.s. learn the state this is not an issue.
- But also need to believe other is sufficiently convinced. And that other is sufficiently convinced that they are sufficiently convinced. etc.
- Choosing action A is optimal for a trader in some period t only if the trader assigns probability at least q to the *joint* event that the state is A and the other chooses A too. (which will depend on their beliefs about a symmetric event)

Is the attack ever carried out? When does individual learning imply common learning?

Trivial case: public signals (perfect correlation). Anything else?

- Discrete time $t = 0, 1, 2, \ldots$
- Common Prior Assumption: $\theta \in \Theta$, Θ finite, according to (prior distribution) p.
- Two players; $\ell, \hat{\ell} \in \{1, 2\}$; results hold for arbitrary finite number of agents.
- Signal Process: $\xi^\theta \equiv \{\xi_t^\theta\}_{t=0}^\infty \equiv \{\xi_{1t}^\theta, \xi_{2t}^\theta\}_{t=0}^\infty$, conditional on θ. ξ^θ iid across t.
  ξ_t^θ takes values $z_t = (z_{1t}, z_{2t}) \in Z_1 \times Z_2 =: Z$.

- States $\Omega \equiv \Theta \times Z^{\infty}$; state ω: parameter and sequence signal profiles
- $\mathbb{P}$ measure on Ω induced by prior p and signal processes $\left(\xi^{\theta}\right)_{\theta \in \Theta}$

  $\mathbb{E}[\cdot]$ expectation wrt $\mathbb{P}$
- $\mathbb{P}^{\theta}$ measure p conditional on $\theta \in \Theta$; $\mathbb{E}^{\theta}[\cdot]$ expectation wrt $\mathbb{P}^{\theta}$
- Period-t history for agent ℓ: $h_{\ell t} \equiv (z_{\ell 0}, z_{\ell 1}, ..., z_{\ell t-1})$

  $H_{\ell t} \equiv (Z_{\ell})^{t}$ space period-t histories for ℓ

  $\{\mathcal{H}_{\ell t}\}_{t=0}^{\infty}$ filtration induced on Ω by agent ℓ's histories
- Filtration $\{\mathcal{F}_t\}_{t \geq 0}$

  Given measurable space $(\Omega, \mathcal{F})$, filtration is a sequence of (sub)σ-algebras $\{\mathcal{F}_t\}_{t \geq 0}$ s.t. $\mathcal{F} \subset \mathcal{F}_t$ $\forall t$ and increasing wrt set inclusion, $\mathcal{F}_t \subset \mathcal{F}_{t'}$ $\forall t \leq t'$

  Filtered Prob. Space or Stochastic Basis is a Prob. Space + Filtration of its σ-algebra

Overview

Player $i \in \{1, ..., I\}$ with knowledge operator K_i.

Definition

A **knowledge hierarchy** among players at ω

Overview

1. Knowledge
2. Common Prior Assumption and Its Implications
3. Common Belief
4. Common Learning
5. Universal Type Space
6. Syntactic Knowledge

Syntactic Knowledge

In the words of Aumann (1999 IJGT, I)

> *One question that often arises is, what do the players know about the [semantic] formalism itself? Does each know the others' partitions? If so, from where does this knowledge derive? If not, how can the formalism indicate what each player knows about the others' knowledge?*
> *(...) More generally, the whole idea of "state of the world," and of a partition structure that accurately reflects the players' knowledge about other players' knowledge, is not transparent. What are the states? Can they be explicitly described? Where do they come from? Where do the information partitions come from?*

Syntactic Knowledge Model

Main Ingredients

Symbols: including Letters from an alphabet $\mathcal{X} := \{x, y, z, ...\}$ taken as fixed, and $\vee, \neg$, (), and κ_i.

Formula (or Propositions): finite string of symbols.

1. Every letter is a formula.
2. If f and g are formulas, so is $(f) \vee (g)$.
3. If f is a formula, so are $\neg(f)$ and $\kappa_i(f)$ for each i.

Interpretation

$\kappa_i f$: "i knows f".

$\vee, \neg$: 'or', 'it is not true that'.

Formula f: a finite concatenation of natural occurrences, using operators and connectives of propositional logic plus the knowledge operators.

Lists of Formulas

Lists

$f \implies g$ means $(\neg f) \vee g$.

$f \iff g$ means $f \implies g$ and $g \implies f$.

List is set of formulas.

Properties of List $\mathcal{L}$

- logically closed if ($f \in \mathcal{L}$ and $f \implies g \in \mathcal{L}$) implies $g \in \mathcal{L}$.
- epistemically closed if $f \in \mathcal{L}$ implies $\kappa_i f \in \mathcal{L}$.
- strongly closed if logically and epistemically closed.
  Strong closure of $\mathcal{L}$ is smalles strongly clost list that includes $\mathcal{L}$.
- coherent if $\neg f \in \mathcal{L}$ implies $f \notin \mathcal{L}$.
- complete if $f \notin \mathcal{L}$ implies $\neg f \in \mathcal{L}$.

Tautologies

Tautology is statement commonly believed by everyone. Formally:

Tautology: a formula in strong closure of the list of all formulas having one of the following forms:

(i) $(f \vee f) \implies f$.

(ii) $f \implies (f \vee g)$.

(iii) $(g \vee f) \implies (f \vee g)$.

(iv) $(f \implies g) \implies ((h \vee f) \implies (h \vee g))$.

(v) $\kappa_i f \implies f$.

(vi) $\kappa_i(f \implies g) \implies ((\kappa_i f) \implies (\kappa_i g))$.

(vii) $\neg\kappa_i f \implies \kappa_i \neg\kappa_i f$.

$\mathcal{T}$: list of all tautologies.

(Theorems are tautologies!)

g is consequence of f if $f \implies g$ is a tautology.

Syntax $\mathcal{S}$: set of all formulas with given population I and alphabet $\mathcal{X}$, countable.

Towards an Isomorphism for Syntactic-Semantic Knowledge

Canonical Semantic Knowledge System: For given syntax $\mathcal{S}$,

State ω:= a closed, coherent, complete list of formulas containing all tautologies.

Ω: set of all states.

Information function: $k_i : \Omega \to 2^\Omega$ s.t. $k_i(\omega)$ is set of formulas in ω starting with κ_i.

Events $E_f := \{\omega \in \Omega : f \in \omega\}$.

For any list $\mathcal{L}$, let $\mathcal{L}^*$ denote the strong closure of $\mathcal{L} \cup \mathcal{T}$.

$\mathcal{L}$ is consistent if $f \in \mathcal{L}^*$ implies $\neg f \notin \mathcal{L}^*$.

Proposition: A list is a state iff it is complete and consistent.

Syntactic and semantic approaches isomorphic (under conditions)

– Aumann (1999 IJGT I, §9).

So What?

Contractual complexity, intrincate financial derivatives, and failures of contigent reasoning: issues in dealing with logical deduction (costly, confusing).

Vague and complex contracts: e.g., Jakobsen (2020 AER), Piermont (2024 wp).

Expanding state space via introducing new concepts/possibilities.

(Fun fact: modern concept of concept originated mainly in Kant's work in 18th century.)