Stopping and Choosing

Duarte Gonçalves

University College London

Topics in Economic Theory

Overview

Overview

1. Stopping: Searching for a Job
   - Job Search
   - Job Search with Discounting
2. Optimal Stopping: Existence and Regularity
3. Satisficing
4. Simple Stopping Rules and Monotone Problems
5. Stopping and Choosing: Selling a House
6. Diamond's Paradox
7. References

Job Search

Accept offer Y_t, continue searching with a per period cost of c.

Interpretation:

- Job search (McCall 1970 QJE): TIOLI salary offers Y_t, cost to search c.
- Selling a house/asset: TIOLI offers Y_t, council tax/management fees c.

$Y_t \sim F$, iid; F continuous, strictly increasing.

Assume $\mathbb{E}[\mathbf{1}_{Y_t \geq 0} Y_t] < \infty$; $Y_0 = 0$; $\mathbb{P}(Y_t > c) > 0$.

Job Search

Accept and get Y_t (present value of getting same wage forever);

Refuse and get z and face same problem tomorrow

Markov problem; state variable = Y_t

Job Search

Accept and get Y_t (present value of getting same wage forever);
Refuse and get z and face same problem tomorrow

Markov problem; state variable = Y_t

Set up Bellman equation; $V(Y_t) = \max\{Y_t, \mathbb{E}[V(Y_{t+1})] - c\}$
(iid $\implies$ stationary problem)

Value: $V(Y_t)$
(handwavy: this presumes a solution and we don't know yet if/why we can do this)

Define $V_t := V(Y_t)$ and $\bar{V} = \mathbb{E}[V(Y_t)]$

Define $\tilde{y} := \bar{V} - c$

Job Search

Accept and get Y_t (present value of getting same wage forever);
Refuse and get z and face same problem tomorrow

Markov problem; state variable = Y_t

Set up Bellman equation; $V(Y_t) = \max\{Y_t, \mathbb{E}[V(Y_{t+1})] - c\}$
(iid $\implies$ stationary problem)

Value: $V(Y_t)$
(handwavy: this presumes a solution and we don't know yet if/why we can do this)

Define $V_t := V(Y_t)$ and $\bar{V} = \mathbb{E}[V(Y_t)]$

Define $\tilde{y} := \bar{V} - c$

Take expectations and get $\tilde{y} + c = \mathbb{E}[\max\{Y_t, \tilde{y}\}] \iff c = \mathbb{E}[(Y_t - \tilde{y})^+] = \int_{\tilde{y}}^{\infty} y \, \mathrm{d}F(y)$

F continuous and strictly increasing: $\exists! \tilde{y} : c = \mathbb{E}[(Y_t - \tilde{y})^+]$

$\tilde{y}$: reservation value

Optimal rule: continue if and only if $Y_t < \tilde{y}$

Job Search with Discounting

Accept offer Y_t, continue searching and receive z; discount factor $\beta \in (0, 1)$.

Interpretation:

- Job search: TIOLI salary offers Y_t, unemployment subsidy z, cost of time β.
- Selling a house/asset: TIOLI offers Y_t, rent acrued z, interest rate r, discount factor $\beta = (1 + r)^{-1}$.

$Y_t \sim F$, iid; F continuous, strictly increasing.

Assume $\mathbb{E}[1_{Y_t \geq 0} Y_t] < \infty$; $Y_0 = 0$; $\mathbb{P}(Y_t > c) > 0$.

Job Search with Discounting

Define $\hat{Y}_t := \frac{\beta^t}{1-\beta} Y_t$ (present value).

Accept and get Y_t forever $\equiv$ Accept and get $\hat{Y}_t$

Refuse, get z, and face same problem tomorrow but discounted by β.

Markov problem; state variable = $\hat{Y}_t$

Job Search with Discounting

Define $\hat{Y}_t := \frac{\beta^t}{1-\beta} Y_t$ (present value).

Accept and get Y_t forever $\equiv$ Accept and get $\hat{Y}_t$

Refuse, get z, and face same problem tomorrow but discounted by β.

Markov problem; state variable = $\hat{Y}_t$

Set up Bellman equation; $V(\hat{Y}_t) = \max\{\hat{Y}_t, z + \beta \mathbb{E}[V(\hat{Y}_{t+1})]\}$

Value: $V(\hat{Y}_t)$

Brief refresher...

Refresher

Definition

$T : X \to X$ is a contraction on (X, d) if $\exists \delta \in [0, 1) : d(T(x), T(y)) \leq \delta d(x, y) \; \forall x, y \in X$.

Refresher

Definition

$T : X \to X$ is a contraction on (X, d) if $\exists \delta \in [0, 1) : d(T(x), T(y)) \leq \delta d(x, y)\ \forall x, y \in X$.

Banach Fixed-Point Theorem

Let (X, d) be a non-empty complete metric space and T a contraction mapping on (X, d). Then, $\exists! x^* \in X : T(x^*) = x^*$. Moreover, for any $x_0 \in X$, $x^* = \lim_{n\to\infty} T^n(x_0)$, where $T^{n+1} := T \circ T^n$ and $T^1 := T$.

Refresher

Definition

$T : X \to X$ is a contraction on (X, d) if $\exists \delta \in [0, 1) : d(T(x), T(y)) \leq \delta d(x, y)\ \forall x, y \in X$.

Banach Fixed-Point Theorem

Let (X, d) be a non-empty complete metric space and T a contraction mapping on (X, d). Then, $\exists! x^* \in X : T(x^*) = x^*$. Moreover, for any $x_0 \in X$, $x^* = \lim_{n\to\infty} T^n(x_0)$, where $T^{n+1} := T \circ T^n$ and $T^1 := T$.

Proof

Let $x_n := T^n(x_0)$. Then $d(x_{n+1}, x_n) = d(T^n(x_1), T^n(x_0)) \leq \delta^n d(x_1, x_0)$, hence $\{x_n\}_n$ is a Cauchy sequence.

(X, d) complete $\equiv$ Cauchy sequences converge $\implies$ x_n converges to some $x^* = T(x^*)$.

Take any $y_0 \in X \setminus \{x_0\}$; define $y_n := T^n(y_0)$; $y_n \to y^*$.

If $x^* \neq y^*$, then $d(y^*, x^*) = d(T^n(y^*), T^n(x^*)) = \delta^n d(y^*, x^*) < d(y^*, x^*)$, a contradiction.

Refresher

Blackwell's Conditions for Contraction Mapping

Let $B(X)$ denote the set of bounded real functions on some nonempty set X endowed with the sup-metric d_∞. Suppose $T : B(X) \to B(X)$ satisfies (i) $\forall f, g \in B(X) : f \geq g \implies T(f) \geq T(g)$, and (ii) $\exists \delta \in [0, 1)$ s.t. $T(f + \alpha) \leq T(f) + \delta\alpha \; \forall f \in B(X)$ and $\forall \alpha \in \mathbb{R}_+$. Then T is a contraction.

Refresher

Blackwell's Conditions for Contraction Mapping

Let $B(X)$ denote the set of bounded real functions on some nonempty set X endowed with the sup-metric d_∞. Suppose $T : B(X) \to B(X)$ satisfies (i) $\forall f, g \in B(X) : f \geq g \implies T(f) \geq T(g)$, and (ii) $\exists \delta \in [0, 1)$ s.t. $T(f + \alpha) \leq T(f) + \delta\alpha \; \forall f \in B(X)$ and $\forall \alpha \in \mathbb{R}_+$. Then T is a contraction.

Proof

For any $f, g \in B(X)$ and $x \in X$, $f(x) - g(x) \leq |f(x) - g(x)| \leq d_\infty(f, g)$.
(i) and (ii): $f \leq g + d_\infty(f, g) \implies T(f) \leq T(g) + \delta d_\infty(f, g)$
and, symmetrically, $T(g) \leq T(f) + \delta d_\infty(f, g)$.
This implies $d_\infty(T(f), T(g)) \leq \delta d_\infty(f, g)$.

Job Search with Discounting

Define $\hat{Y}_t := \frac{\beta^t}{1-\beta} Y_t$ (present value).

Accept and get Y_t forever $\equiv$ Accept and get $\hat{Y}_t$

Refuse, get z, and face same problem tomorrow but discounted by β.

Markov problem; state variable = $\hat{Y}_t$

Set up Bellman equation; $V(\hat{Y}_t) = \max\{\hat{Y}_t, z + \beta\mathbb{E}[V(\hat{Y}_{t+1})]\}$

Value: $V(\hat{Y}_t)$, well-defined

Define $V_t := V(\hat{Y}_t)$ and $\bar{V} = \mathbb{E}[V(\hat{Y}_t)]$

Job Search with Discounting

Define $\hat{Y}_t := \frac{\beta^t}{1-\beta} Y_t$ (present value).

Accept and get Y_t forever $\equiv$ Accept and get $\hat{Y}_t$

Refuse, get z, and face same problem tomorrow but discounted by β.

Markov problem; state variable = $\hat{Y}_t$

Set up Bellman equation; $V(\hat{Y}_t) = \max\{\hat{Y}_t, z + \beta \mathbb{E}[V(\hat{Y}_{t+1})]\}$

Value: $V(\hat{Y}_t)$, well-defined

Define $V_t := V(\hat{Y}_t)$ and $\bar{V} = \mathbb{E}[V(\hat{Y}_t)]$

Take expectations and get

$$\bar{V} = \mathbb{E}[\max\{\hat{Y}_t, z + \beta\bar{V}\}] \iff \bar{V}(1-\beta) = z + \mathbb{E}[(\hat{Y}_t - (z + \beta\bar{V}))^+] = \int_{z+\beta\bar{V}}^{\infty} \frac{1}{1-\beta} y \,\mathrm{d}F(y)$$

F continuous: $\exists! \bar{V} : \bar{V}(1-\beta) = z + \mathbb{E}[(\hat{Y}_t - (z + \beta\bar{V}))^+]$

$\tilde{y} := (1-\beta)(z + \beta\bar{V})$: reservation value

Optimal rule: continue if and only if $Y_t < \tilde{y}$

Overview

Going Beyond the Basic Setting

Y_t may not be iid

- Depend on time of unemployment
- Result from underlying dynamic game between recruiting firms
- Uncertain market conditions (hence perception of F evolves over time depending on past Y_ℓ)

...

Going Beyond the Basic Setting

Y_t may not be iid

- Depend on time of unemployment
- Result from underlying dynamic game between recruiting firms
- Uncertain market conditions (hence perception of F evolves over time depending on past Y_ℓ)

...

Introduce general tools to tackle the problem

Setup and Assumptions

$\{X_0, X_1, X_2, ...\}$ rv whose joint distribution is assumed to be known; write $X^t := (X_\ell)_{\ell=1,...,t}$.

Sequence of functions $x^t \mapsto y_t(x^t) \in \mathbb{R}$; write $Y_t := y_t(x^t)$.

Filtration $\mathbb{F} = \{\mathcal{F}_t\} = \sigma(X^t)$.

Adapted payoff process $\{Y_t\}$; terminal Y_∞ (possibly $-\infty$).

Stopping time τ: $\{\tau \leq t\} \in \mathcal{F}_t$ for all t; feasible set $\mathbb{T}$.

Objective: maximise value of Y by adequately choosing stopping time, $\sup_{\tau \in \mathbb{T}} \mathbb{E}[Y_\tau]$.

Setup and Assumptions

$\{X_0, X_1, X_2, ...\}$ rv whose joint distribution is assumed to be known; write $X^t := (X_\ell)_{\ell=1,...,t}$.

Sequence of functions $x^t \mapsto y_t(x^t) \in \mathbb{R}$; write $Y_t := y_t(x^t)$.

Filtration $\mathbb{F} = \{\mathcal{F}_t\} = \sigma(X^t)$.

Adapted payoff process $\{Y_t\}$; terminal Y_∞ (possibly $-\infty$).

Stopping time τ: $\{\tau \leq t\} \in \mathcal{F}_t$ for all t; feasible set $\mathbb{T}$.

Objective: maximise value of Y by adequately choosing stopping time, $\sup_{\tau \in \mathbb{T}} \mathbb{E}[Y_\tau]$.

Two questions:

1. When is there actually an optimal stopping time? (Is sup actually a max?)
2. If so, what does it look like?

   Previous applications: guess and verify or use specific structural assumptions.
   Now: use very general assumptions.

Setup and Assumptions

$\{X_0, X_1, X_2, ...\}$ rv whose joint distribution is assumed to be known; write $X^t := (X_\ell)_{\ell=1,...,t}$.

Sequence of functions $x^t \mapsto y_t(x^t) \in \mathbb{R}$; write $Y_t := y_t(x^t)$.

Filtration $\mathbb{F} = \{\mathcal{F}_t\} = \sigma(X^t)$.

Adapted payoff process $\{Y_t\}$; terminal Y_∞ (possibly $-\infty$).

Stopping time τ: $\{\tau \leq t\} \in \mathcal{F}_t$ for all t; feasible set $\mathbb{T}$.

Objective: maximise value of Y by adequately choosing stopping time, $\sup_{\tau \in \mathbb{T}} \mathbb{E}[Y_\tau]$.

Two questions:

1. When is there actually an optimal stopping time? (Is sup actually a max?)
2. If so, what does it look like?

   Previous applications: guess and verify or use specific structural assumptions.
   Now: use very general assumptions.

Standing assumptions

(A1) $\mathbb{E}\left[\sup_{t\geq 0} Y_t\right] < \infty$.

(A2) $\lim_{t\to\infty} \mathbb{E}[Y_t] \leq Y_\infty$ a.s.

Note: (A1) implies $\sup_\tau \mathbb{E}[Y_\tau] < \infty$

Regular Stopping Times

Definition (Regularity)

τ is regular if for all t, $\mathbb{E}[Y_\tau \mid \mathcal{F}_t] > Y_t$ a.s. on $\{\tau > t\}$.

Regular Stopping Times

Definition (Regularity)

τ is regular if for all t, $\mathbb{E}[Y_\tau \mid \mathcal{F}_t] > Y_t$ a.s. on $\{\tau > t\}$.

Lemma 1 (Regularity is wloo)

Under (A1), for any stopping time τ there exists a *regular* stopping time $\rho \leq \tau$ with $\mathbb{E}[Y_\rho] \geq \mathbb{E}[Y_\tau]$.

Regular Stopping Times

Definition (Regularity)

τ is regular if for all t, $\mathbb{E}[Y_\tau \mid \mathcal{F}_t] > Y_t$ a.s. on $\{\tau > t\}$.

Lemma 1 (Regularity is wloo)

Under (A1), for any stopping time τ there exists a *regular* stopping time $\rho \leq \tau$ with $\mathbb{E}[Y_\rho] \geq \mathbb{E}[Y_\tau]$.

Lemma 2 (Regularity is closed under $\vee$)

Under (A1), if τ and ρ are regular, then $\xi := \tau \vee \rho$ is regular and $\mathbb{E}[Y_\xi] \geq \max\{\mathbb{E}[Y_\tau], \mathbb{E}[Y_\rho]\}$.

Proof of Lemma 1 (Regularity wloo)

Proof

Fix τ with $\mathbb{E}[|Y_\tau|] < \infty$ (true by (A1) since $Y_\tau \le \sup_s Y_s$).

Proof of Lemma 1 (Regularity wloo)

Proof

Fix τ with $\mathbb{E}[|Y_\tau|] < \infty$ (true by (A1) since $Y_\tau \leq \sup_s Y_s$).

Define $Z_t := \mathbb{E}[Y_\tau \mid \mathcal{F}_t]$ and let $\rho := \inf\{t \geq \mathbf{0} : Z_t \leq Y_t\}$.

On $\{\rho > t\}$: $Y_t < Z_t = \mathbb{E}[Y_\tau \mid \mathcal{F}_t]$, so ρ is regular.

Proof of Lemma 1 (Regularity wloo)

Proof

Fix τ with $\mathbb{E}[|Y_\tau|] < \infty$ (true by (A1) since $Y_\tau \leq \sup_s Y_s$).

Define $Z_t := \mathbb{E}[Y_\tau \mid \mathcal{F}_t]$ and let $\rho := \inf\{t \geq \mathbf{0} : Z_t \leq Y_t\}$.

On $\{\rho > t\}$: $Y_t < Z_t = \mathbb{E}[Y_\tau \mid \mathcal{F}_t]$, so ρ is regular.

On $\{\rho = t\}$: $Y_\rho = Y_t \geq Z_t = \mathbb{E}[Y_\tau \mid \mathcal{F}_t]$. On $\{\rho = \infty\}$: $Y_\rho = Y_\infty = Y_\tau$ a.s.

Proof of Lemma 1 (Regularity wloo)

Proof

Fix τ with $\mathbb{E}[|Y_\tau|] < \infty$ (true by (A1) since $Y_\tau \leq \sup_s Y_s$).

Define $Z_t := \mathbb{E}[Y_\tau \mid \mathcal{F}_t]$ and let $\rho := \inf\{t \geq 0 : Z_t \leq Y_t\}$.

On $\{\rho > t\}$: $Y_t < Z_t = \mathbb{E}[Y_\tau \mid \mathcal{F}_t]$, so ρ is regular.

On $\{\rho = t\}$: $Y_\rho = Y_t \geq Z_t = \mathbb{E}[Y_\tau \mid \mathcal{F}_t]$. On $\{\rho = \infty\}$: $Y_\rho = Y_\infty = Y_\tau$ a.s.

Hence

$$\mathbb{E}[Y_\rho] = \sum_{t=0}^{\infty} \mathbb{E}[\mathbf{1}_{\{\rho=t\}} Y_t] + \mathbb{E}[\mathbf{1}_{\{\rho=\infty\}} Y_\infty]$$

Proof of Lemma 1 (Regularity wloo)

Proof

Fix τ with $\mathbb{E}[|Y_\tau|] < \infty$ (true by (A1) since $Y_\tau \leq \sup_s Y_s$).

Define $Z_t := \mathbb{E}[Y_\tau \mid \mathcal{F}_t]$ and let $\rho := \inf\{t \geq 0 : Z_t \leq Y_t\}$.

On $\{\rho > t\}$: $Y_t < Z_t = \mathbb{E}[Y_\tau \mid \mathcal{F}_t]$, so ρ is regular.

On $\{\rho = t\}$: $Y_\rho = Y_t \geq Z_t = \mathbb{E}[Y_\tau \mid \mathcal{F}_t]$. On $\{\rho = \infty\}$: $Y_\rho = Y_\infty = Y_\tau$ a.s.

Hence

$$\begin{aligned}\mathbb{E}[Y_\rho] &= \sum_{t=0}^{\infty} \mathbb{E}[\mathbf{1}_{\{\rho=t\}} Y_t] + \mathbb{E}[\mathbf{1}_{\{\rho=\infty\}} Y_\infty] \\ &\geq \sum_{t=0}^{\infty} \mathbb{E}[\mathbf{1}_{\{\rho=t\}} \mathbb{E}[Y_\tau \mid \mathcal{F}_t]] + \mathbb{E}[\mathbf{1}_{\{\rho=\infty\}} Y_\tau]\end{aligned}$$

Proof of Lemma 1 (Regularity wloo)

Proof

Fix τ with $\mathbb{E}[|Y_\tau|] < \infty$ (true by (A1) since $Y_\tau \le \sup_s Y_s$).

Define $Z_t := \mathbb{E}[Y_\tau \mid \mathcal{F}_t]$ and let $\rho := \inf\{t \ge 0 : Z_t \le Y_t\}$.

On $\{\rho > t\}$: $Y_t < Z_t = \mathbb{E}[Y_\tau \mid \mathcal{F}_t]$, so ρ is regular.

On $\{\rho = t\}$: $Y_\rho = Y_t \ge Z_t = \mathbb{E}[Y_\tau \mid \mathcal{F}_t]$. On $\{\rho = \infty\}$: $Y_\rho = Y_\infty = Y_\tau$ a.s.

Hence

$$
\begin{aligned}
\mathbb{E}[Y_\rho] &= \sum_{t=0}^{\infty} \mathbb{E}[\mathbf{1}_{\{\rho=t\}} Y_t] + \mathbb{E}[\mathbf{1}_{\{\rho=\infty\}} Y_\infty] \\
&\ge \sum_{t=0}^{\infty} \mathbb{E}[\mathbf{1}_{\{\rho=t\}} \mathbb{E}[Y_\tau \mid \mathcal{F}_t]] + \mathbb{E}[\mathbf{1}_{\{\rho=\infty\}} Y_\tau] \\
&\ge \sum_{t=0}^{\infty} \mathbb{E}[\mathbf{1}_{\{\rho=t\}} Y_\tau] + \mathbb{E}[\mathbf{1}_{\{\rho=\infty\}} Y_\tau] \\
&= \mathbb{E}[Y_\tau].
\end{aligned}
$$

Proof of Lemma 1 (Regularity wloo)

Proof

Fix τ with $\mathbb{E}[|Y_\tau|] < \infty$ (true by (A1) since $Y_\tau \leq \sup_s Y_s$).

Define $Z_t := \mathbb{E}[Y_\tau \mid \mathcal{F}_t]$ and let $\rho := \inf\{t \geq 0 : Z_t \leq Y_t\}$.

On $\{\rho > t\}$: $Y_t < Z_t = \mathbb{E}[Y_\tau \mid \mathcal{F}_t]$, so ρ is regular.

On $\{\rho = t\}$: $Y_\rho = Y_t \geq Z_t = \mathbb{E}[Y_\tau \mid \mathcal{F}_t]$. On $\{\rho = \infty\}$: $Y_\rho = Y_\infty = Y_\tau$ a.s.

Hence

$$\begin{aligned}
\mathbb{E}[Y_\rho] &= \sum_{t=0}^{\infty} \mathbb{E}[\mathbf{1}_{\{\rho=t\}} Y_t] + \mathbb{E}[\mathbf{1}_{\{\rho=\infty\}} Y_\infty] \\
&\geq \sum_{t=0}^{\infty} \mathbb{E}[\mathbf{1}_{\{\rho=t\}} \mathbb{E}[Y_\tau \mid \mathcal{F}_t]] + \mathbb{E}[\mathbf{1}_{\{\rho=\infty\}} Y_\tau] \\
&\geq \sum_{t=0}^{\infty} \mathbb{E}[\mathbf{1}_{\{\rho=t\}} Y_\tau] + \mathbb{E}[\mathbf{1}_{\{\rho=\infty\}} Y_\tau] \\
&= \mathbb{E}[Y_\tau].
\end{aligned}$$

Suppose $\neg(\rho \leq \tau)$; note that, at $\{\rho > \tau = t\}$, $Z_t = Z_\tau = Y_\tau < Z_t$, a contradiction.

Proof of Lemma 2 (Regularity is closed under $\vee$)

Proof

1. Proving ξ is regular:

   $\{\xi > t\} = \{\xi = \tau > t\} \cup \{\xi = \rho > t\}.$

Proof of Lemma 2 (Regularity is closed under $\vee$)

Proof

1. Proving ξ is regular:

   $\{\xi > t\} = \{\xi = \tau > t\} \cup \{\xi = \rho > t\}$.

   On $\{\xi = \tau > t\}$, $\mathbb{E}[Y_\xi \mid \mathcal{F}_t] = \mathbb{E}[Y_\tau \mid \mathcal{F}_t] > Y_t$ a.s. $\because$ τ is regular.

Proof of Lemma 2 (Regularity is closed under $\vee$)

Proof

1. Proving ξ is regular:

   $\{\xi > t\} = \{\xi = \tau > t\} \cup \{\xi = \rho > t\}$.

   On $\{\xi = \tau > t\}$, $\mathbb{E}[Y_\xi \mid \mathcal{F}_t] = \mathbb{E}[Y_\tau \mid \mathcal{F}_t] > Y_t$ a.s. $\because$ τ is regular.

   Symmetrically, on $\{\xi = \rho > t\}$, $\mathbb{E}[Y_\xi \mid \mathcal{F}_t] = \mathbb{E}[Y_\rho \mid \mathcal{F}_t] > Y_t$ a.s. $\because$ ρ is regular.

Proof of Lemma 2 (Regularity is closed under $\vee$)

Proof

1. Proving ξ is regular:

   $\{\xi > t\} = \{\xi = \tau > t\} \cup \{\xi = \rho > t\}$.

   On $\{\xi = \tau > t\}$, $\mathbb{E}[Y_\xi \mid \mathcal{F}_t] = \mathbb{E}[Y_\tau \mid \mathcal{F}_t] > Y_t$ a.s. $\because$ τ is regular.

   Symmetrically, on $\{\xi = \rho > t\}$, $\mathbb{E}[Y_\xi \mid \mathcal{F}_t] = \mathbb{E}[Y_\rho \mid \mathcal{F}_t] > Y_t$ a.s. $\because$ ρ is regular.

2. Proving $\mathbb{E}[Y_\xi] \geq \mathbb{E}[Y_\tau] \vee \mathbb{E}[Y_\rho]$:

   On $\{\xi = \tau = t\}$, $Y_\xi = Y_\tau = Y_t$.

Proof of Lemma 2 (Regularity is closed under $\vee$)

Proof

1. Proving ξ is regular:

   $\{\xi > t\} = \{\xi = \tau > t\} \cup \{\xi = \rho > t\}$.

   On $\{\xi = \tau > t\}$, $\mathbb{E}[Y_\xi \mid \mathcal{F}_t] = \mathbb{E}[Y_\tau \mid \mathcal{F}_t] > Y_t$ a.s. $\because$ τ is regular.

   Symmetrically, on $\{\xi = \rho > t\}$, $\mathbb{E}[Y_\xi \mid \mathcal{F}_t] = \mathbb{E}[Y_\rho \mid \mathcal{F}_t] > Y_t$ a.s. $\because$ ρ is regular.

2. Proving $\mathbb{E}[Y_\xi] \geq \mathbb{E}[Y_\tau] \vee \mathbb{E}[Y_\rho]$:

   On $\{\xi = \tau = t\}$, $Y_\xi = Y_\tau = Y_t$.

   On $\{\xi = \rho > \tau = t\}$, $\xi = \rho$ and $\mathbb{E}[Y_\xi \mid \mathcal{F}_t] = \mathbb{E}[Y_\rho \mid \mathcal{F}_t] > Y_t = Y_\tau$ a.s.

Proof of Lemma 2 (Regularity is closed under $\vee$)

Proof

1. Proving ξ is regular:

   $\{\xi > t\} = \{\xi = \tau > t\} \cup \{\xi = \rho > t\}$.

   On $\{\xi = \tau > t\}$, $\mathbb{E}[Y_\xi \mid \mathcal{F}_t] = \mathbb{E}[Y_\tau \mid \mathcal{F}_t] > Y_t$ a.s. $\because$ τ is regular.

   Symmetrically, on $\{\xi = \rho > t\}$, $\mathbb{E}[Y_\xi \mid \mathcal{F}_t] = \mathbb{E}[Y_\rho \mid \mathcal{F}_t] > Y_t$ a.s. $\because$ ρ is regular.

2. Proving $\mathbb{E}[Y_\xi] \geq \mathbb{E}[Y_\tau] \vee \mathbb{E}[Y_\rho]$:

   On $\{\xi = \tau = t\}$, $Y_\xi = Y_\tau = Y_t$.

   On $\{\xi = \rho > \tau = t\}$, $\xi = \rho$ and $\mathbb{E}[Y_\xi \mid \mathcal{F}_t] = \mathbb{E}[Y_\rho \mid \mathcal{F}_t] > Y_t = Y_\tau$ a.s.

   Hence

$$\mathbb{E}[Y_\xi] = \sum_{t=0}^{\infty} \mathbb{E}[\mathbf{1}_{\{\tau=t\}} Y_\xi] + \mathbb{E}[\mathbf{1}_{\{\tau=\infty\}} Y_\xi] = \sum_{t=0}^{\infty} \mathbb{E}[\mathbf{1}_{\{\tau=t\}} \mathbb{E}[Y_\xi \mid \mathcal{F}_t]] + \mathbb{E}[\mathbf{1}_{\{\tau=\infty\}} Y_\xi]$$

$$\geq \sum_{t=0}^{\infty} \mathbb{E}[\mathbf{1}_{\{\tau=t\}} Y_\tau] + \mathbb{E}[\mathbf{1}_{\{\tau=\infty\}} Y_\tau] = \mathbb{E}[Y_\tau].$$

Proof of Lemma 2 (Regularity is closed under $\vee$)

Proof

1. Proving ξ is regular:

   $\{\xi > t\} = \{\xi = \tau > t\} \cup \{\xi = \rho > t\}$.

   On $\{\xi = \tau > t\}$, $\mathbb{E}[Y_\xi \mid \mathcal{F}_t] = \mathbb{E}[Y_\tau \mid \mathcal{F}_t] > Y_t$ a.s. $\because$ τ is regular.

   Symmetrically, on $\{\xi = \rho > t\}$, $\mathbb{E}[Y_\xi \mid \mathcal{F}_t] = \mathbb{E}[Y_\rho \mid \mathcal{F}_t] > Y_t$ a.s. $\because$ ρ is regular.

2. Proving $\mathbb{E}[Y_\xi] \geq \mathbb{E}[Y_\tau] \vee \mathbb{E}[Y_\rho]$:

   On $\{\xi = \tau = t\}$, $Y_\xi = Y_\tau = Y_t$.

   On $\{\xi = \rho > \tau = t\}$, $\xi = \rho$ and $\mathbb{E}[Y_\xi \mid \mathcal{F}_t] = \mathbb{E}[Y_\rho \mid \mathcal{F}_t] > Y_t = Y_\tau$ a.s.

   Hence

$$\mathbb{E}[Y_\xi] = \sum_{t=0}^{\infty} \mathbb{E}[\mathbf{1}_{\{\tau=t\}} Y_\xi] + \mathbb{E}[\mathbf{1}_{\{\tau=\infty\}} Y_\xi] = \sum_{t=0}^{\infty} \mathbb{E}[\mathbf{1}_{\{\tau=t\}} \mathbb{E}[Y_\xi \mid \mathcal{F}_t]] + \mathbb{E}[\mathbf{1}_{\{\tau=\infty\}} Y_\xi]$$

$$\geq \sum_{t=0}^{\infty} \mathbb{E}[\mathbf{1}_{\{\tau=t\}} Y_\tau] + \mathbb{E}[\mathbf{1}_{\{\tau=\infty\}} Y_\tau] = \mathbb{E}[Y_\tau].$$

   By a symmetric argument, $\mathbb{E}[Y_\xi] \geq \max\{\mathbb{E}[Y_\tau], \mathbb{E}[Y_\rho]\}$.

Existence

Theorem (Existence)

Under (A1) and (A2), there is a regular τ such that $\mathbb{E}[Y_\tau] = \sup_{\rho \in \mathbb{T}} \mathbb{E}[Y_\rho]$.

Proof

Take the case $V^* := \sup_{\rho \in \mathbb{T}} \mathbb{E}[Y_\rho] > -\infty$.

Existence

Theorem (Existence)

Under (A1) and (A2), there is a regular $\boldsymbol{\tau}$ such that $\mathbb{E}[Y_{\tau}] = \sup_{\rho \in \mathbb{T}} \mathbb{E}[Y_{\rho}]$.

Proof

Take the case $V^* := \sup_{\rho \in \mathbb{T}} \mathbb{E}[Y_{\rho}] > -\infty$.

By definition of sup, $\exists$ sequence $\hat{\tau}_1, \hat{\tau}_2, \ldots$ such that $\mathbb{E}[Y_{\hat{\tau}_n}] \to V^*$.

Existence

Theorem (Existence)

Under (A1) and (A2), there is a regular τ such that $\mathbb{E}[Y_\tau] = \sup_{\rho \in \mathbb{T}} \mathbb{E}[Y_\rho]$.

Proof

Take the case $V^* := \sup_{\rho \in \mathbb{T}} \mathbb{E}[Y_\rho] > -\infty$.

By definition of sup, $\exists$ sequence $\hat{\tau}_1, \hat{\tau}_2, \ldots$ such that $\mathbb{E}[Y_{\hat{\tau}_n}] \to V^*$.

Define regularised $\rho_n := \inf\{t \geq 0 : \mathbb{E}[Y_{\hat{\tau}_n} \mid \mathcal{F}_t] \leq Y_t\}$; let $\tau_n := \max\{\rho_1, \rho_2, \ldots, \rho_n\}$, regular.

Existence

Theorem (Existence)

Under (A1) and (A2), there is a regular τ such that $\mathbb{E}[Y_\tau] = \sup_{\rho \in \mathbb{T}} \mathbb{E}[Y_\rho]$.

Proof

Take the case $V^* := \sup_{\rho \in \mathbb{T}} \mathbb{E}[Y_\rho] > -\infty$.

By definition of sup, $\exists$ sequence $\hat{\tau}_1, \hat{\tau}_2, \ldots$ such that $\mathbb{E}[Y_{\hat{\tau}_n}] \to V^*$.

Define regularised $\rho_n := \inf\{t \geq 0 : \mathbb{E}[Y_{\hat{\tau}_n} \mid \mathcal{F}_t] \leq Y_t\}$; let $\tau_n := \max\{\rho_1, \rho_2, \ldots, \rho_n\}$, regular.

By the lemmas, $V^* \geq \mathbb{E}[Y_{\tau_n}] \geq \max_{\ell=1,\ldots,n} \mathbb{E}[Y_{\rho_n}] \geq \mathbb{E}[Y_{\hat{\tau}_n}] \to V^*$.

Existence

Theorem (Existence)

Under (A1) and (A2), there is a regular τ such that $\mathbb{E}[Y_\tau] = \sup_{\rho \in \mathbb{T}} \mathbb{E}[Y_\rho]$.

Proof

Take the case $V^* := \sup_{\rho \in \mathbb{T}} \mathbb{E}[Y_\rho] > -\infty$.

By definition of sup, $\exists$ sequence $\hat{\tau}_1, \hat{\tau}_2, \ldots$ such that $\mathbb{E}[Y_{\hat{\tau}_n}] \to V^*$.

Define regularised $\rho_n := \inf\{t \geq 0 : \mathbb{E}[Y_{\hat{\tau}_n} \mid \mathcal{F}_t] \leq Y_t\}$; let $\tau_n := \max\{\rho_1, \rho_2, \ldots, \rho_n\}$, regular.

By the lemmas, $V^* \geq \mathbb{E}[Y_{\tau_n}] \geq \max_{\ell=1,\ldots,n} \mathbb{E}[Y_{\rho_n}] \geq \mathbb{E}[Y_{\hat{\tau}_n}] \to V^*$.

Define $\tau_\infty := \sup_{n \in \mathbb{N}} \rho_n$. τ_n pointwise increasing $\implies$ pointwise converges to τ_∞.
Moreover, $\limsup_{n \to \infty} Y_{\tau_n} \leq Y_{\tau_\infty}$ a.s. (from (A2)).

Existence

Theorem (Existence)

Under (A1) and (A2), there is a regular τ such that $\mathbb{E}[Y_\tau] = \sup_{\rho \in \mathbb{T}} \mathbb{E}[Y_\rho]$.

Proof

Take the case $V^* := \sup_{\rho \in \mathbb{T}} \mathbb{E}[Y_\rho] > -\infty$.

By definition of sup, $\exists$ sequence $\hat{\tau}_1, \hat{\tau}_2, \ldots$ such that $\mathbb{E}[Y_{\hat{\tau}_n}] \to V^*$.

Define regularised $\rho_n := \inf\{t \geq 0 : \mathbb{E}[Y_{\hat{\tau}_n} \mid \mathcal{F}_t] \leq Y_t\}$; let $\tau_n := \max\{\rho_1, \rho_2, \ldots, \rho_n\}$, regular.

By the lemmas, $V^* \geq \mathbb{E}[Y_{\tau_n}] \geq \max_{\ell=1,\ldots,n} \mathbb{E}[Y_{\rho_n}] \geq \mathbb{E}[Y_{\hat{\tau}_n}] \to V^*$.

Define $\tau_\infty := \sup_{n \in \mathbb{N}} \rho_n$. τ_n pointwise increasing $\implies$ pointwise converges to τ_∞. Moreover, $\limsup_{n\to\infty} Y_{\tau_n} \leq Y_{\tau_\infty}$ a.s. (from (A2)).

Note: by construction, $\mathbb{E}[\limsup_n Y_{\tau_n}] \leq \mathbb{E}[Y_{\tau_\infty}]$. Since $Y_{\tau_n} \leq \sup_n Y_n$, by Fatou's lemma and (A1), $\mathbb{E}[\liminf_{n\to\infty} \sup_m Y_m - Y_{\tau_n}] \leq \liminf_{n\to\infty} \mathbb{E}[\sup_m Y_m - Y_{\tau_n}] \implies V^* = \limsup_{n\to\infty} \mathbb{E}[Y_{\tau_n}] \leq \mathbb{E}[\limsup_{n\to\infty} Y_{\tau_n}] \leq \mathbb{E}[Y_{\tau_\infty}]$.

Existence

Theorem (Existence)

Under (A1) and (A2), there is a regular τ such that $\mathbb{E}[Y_\tau] = \sup_{\rho \in \mathbb{T}} \mathbb{E}[Y_\rho]$.

Proof

Take the case $V^* := \sup_{\rho \in \mathbb{T}} \mathbb{E}[Y_\rho] > -\infty$.

By definition of sup, $\exists$ sequence $\hat{\tau}_1, \hat{\tau}_2, \ldots$ such that $\mathbb{E}[Y_{\hat{\tau}_n}] \to V^*$.

Define regularised $\rho_n := \inf\{t \geq 0 : \mathbb{E}[Y_{\hat{\tau}_n} \mid \mathcal{F}_t] \leq Y_t\}$; let $\tau_n := \max\{\rho_1, \rho_2, \ldots, \rho_n\}$, regular.

By the lemmas, $V^* \geq \mathbb{E}[Y_{\tau_n}] \geq \max_{\ell=1,\ldots,n} \mathbb{E}[Y_{\rho_n}] \geq \mathbb{E}[Y_{\hat{\tau}_n}] \to V^*$.

Define $\tau_\infty := \sup_{n \in \mathbb{N}} \rho_n$. τ_n pointwise increasing $\implies$ pointwise converges to τ_∞. Moreover, $\limsup_{n\to\infty} Y_{\tau_n} \leq Y_{\tau_\infty}$ a.s. (from (A2)).

Note: by construction, $\mathbb{E}[\limsup_n Y_{\tau_n}] \leq \mathbb{E}[Y_{\tau_\infty}]$. Since $Y_{\tau_n} \leq \sup_n Y_n$, by Fatou's lemma and (A1), $\mathbb{E}[\liminf_{n\to\infty} \sup_m Y_m - Y_{\tau_n}] \leq \liminf_{n\to\infty} \mathbb{E}[\sup_m Y_m - Y_{\tau_n}] \implies V^* = \limsup_{n\to\infty} \mathbb{E}[Y_{\tau_n}] \leq \mathbb{E}[\limsup_{n\to\infty} Y_{\tau_n}] \leq \mathbb{E}[Y_{\tau_\infty}]$.

Conclude: $V^* = \sup_{\rho \in \mathbb{T}} \mathbb{E}[Y_\rho] \geq \mathbb{E}[Y_{\tau_\infty}] \geq V^*$.

Assumptions

Example

Let $X_t \sim$Bernoulli(1/2) iid; $Y_0 := \mathbf{0}$, $Y_t := (\mathbf{2}^t - \mathbf{1}) \prod_{\ell=\mathbf{1}}^{t} X_\ell$ for $t \in \mathbb{N}$, $Y_\infty := \mathbf{0}$.

Assumptions

Example

Let $X_t \sim$Bernoulli(1/2) iid; $Y_0 := 0$, $Y_t := (2^t - 1)\prod_{\ell=1}^t X_\ell$ for $t \in \mathbb{N}$, $Y_\infty := 0$.

Fails (A1): Note $\sup_{k \leq t} Y_k = 2^k - 1$ with probability $2^{-(k+1)}$ for $k = 0, 1, \ldots, t-1$ and with probability 2^{-t} for $k = t$. Hence $\mathbb{E}[\sup_t Y_t] = \sum_{t=0}^\infty (2^t - 1)2^{-(t+1)} = \infty$.

Assumptions

Example

Let $X_t \sim$Bernoulli(1/2) iid; $Y_0 := 0$, $Y_t := (2^t - 1)\prod_{\ell=1}^t X_\ell$ for $t \in \mathbb{N}$, $Y_\infty := 0$.

Fails (A1): Note $\sup_{k \leq t} Y_k = 2^k - 1$ with probability $2^{-(k+1)}$ for $k = 0, 1, \ldots, t-1$ and with probability 2^{-t} for $k = t$. Hence $\mathbb{E}[\sup_t Y_t] = \sum_{t=0}^{\infty}(2^t - 1)2^{-(t+1)} = \infty$.

Satisfies (A2): $Y_t \to 0$ a.s.

Assumptions

Example

Let $X_t \sim$Bernoulli(1/2) iid; $Y_0 := 0$, $Y_t := (2^t - 1)\prod_{\ell=1}^{t} X_\ell$ for $t \in \mathbb{N}$, $Y_\infty := 0$.

Fails (A1): Note $\sup_{k \leq t} Y_k = 2^k - 1$ with probability $2^{-(k+1)}$ for $k = 0, 1, ..., t-1$ and with probability 2^{-t} for $k = t$. Hence $\mathbb{E}[\sup_t Y_t] = \sum_{t=0}^{\infty}(2^t - 1)2^{-(t+1)} = \infty$.

Satisfies (A2): $Y_t \to 0$ a.s.

Indeed, no optimal stopping time. Conditional on reaching t with $Y_t > 0 \iff \prod_{\ell=1}^{t} X_\ell = 1$, then don't want to stop: $Y_t = 2^t - 1 < 2^t - 1/2 = (1/2)(2^{t+1} - 1) = \mathbb{E}[Y_{t+1} | Y_t > 0]$.

Assumptions

Example

Let $Y_0 := 0$, $Y_t := 1 - 1/t$ for $t \in \mathbb{N}$, $Y_\infty := 0$.

Assumptions

Example

Let $Y_0 := 0$, $Y_t := 1 - 1/t$ for $t \in \mathbb{N}$, $Y_\infty := 0$.

Satisfies (A1): $Y_t \leq 1$.

Assumptions

Example

Let $Y_0 := 0$, $Y_t := 1 - 1/t$ for $t \in \mathbb{N}$, $Y_\infty := 0$.

Satisfies (A1): $Y_t \leq 1$.

Fails (A2): $Y_t \to 1 > 0 = Y_\infty$.

Assumptions

Example

Let $Y_0 := 0$, $Y_t := 1 - 1/t$ for $t \in \mathbb{N}$, $Y_\infty := 0$.

Satisfies (A1): $Y_t \leq 1$.

Fails (A2): $Y_t \to 1 > 0 = Y_\infty$.

Indeed, no optimal stopping time as $Y_t < Y_{t+1}$.

Dynamic Programming Principle

Want something like Bellman equation/DPP: stop today or continue assuming optimal stopping from then on

Need some definitions first...

Dynamic Programming Principle

Want something like Bellman equation/DPP: stop today or continue assuming optimal stopping from then on

Need some definitions first...

Definition

Let $(X_t)_{t \in T}$ be a collection of rv. Z rv is essential supremum of $(X_t)_{t \in T}$, $Z = \operatorname{ess\,sup}_{t \in T} X_t$, if (i) $\mathbb{P}(Z \geq X_t) = 1$ $\forall t \in T$ ('probabilistic upper bound'), and (ii) $\forall Z' : \mathbb{P}(Z' \geq X_t) = 1$ $\forall t \in T$, $\mathbb{P}(Z' \geq Z) = 1$ (smallest probabilistic upper bound).

Dynamic Programming Principle

Want something like Bellman equation/DPP: stop today or continue assuming optimal stopping from then on

Need some definitions first...

Definition

Let $(X_t)_{t\in T}$ be a collection of rv. Z rv is essential supremum of $(X_t)_{t\in T}$, $Z = \operatorname{ess\,sup}_{t\in T} X_t$, if (i) $\mathbb{P}(Z \geq X_t) = 1 \ \forall t \in T$ ('probabilistic upper bound'), and (ii) $\forall Z' : \mathbb{P}(Z' \geq X_t) = 1 \ \forall t \in T$, $\mathbb{P}(Z' \geq Z) = 1$ (smallest probabilistic upper bound).

Lemma 3

Let $(X_t)_{t\in T}$ be any collection of rv.
An essential supremum always exists.
Furthermore, $\exists$ a countable $C \subset T : \sup_{t\in C} X_t = \operatorname{ess\,sup}_{t\in T} X_t$.

Dynamic Programming Principle

Want something like Bellman equation/DPP: stop today or continue assuming optimal stopping from then on

Need some definitions first...

Definition

Let $(X_t)_{t\in T}$ be a collection of rv. Z rv is essential supremum of $(X_t)_{t\in T}$, $Z = \operatorname{ess\,sup}_{t\in T} X_t$, if (i) $\mathbb{P}(Z \geq X_t) = 1$ $\forall t \in T$ ('probabilistic upper bound'), and (ii) $\forall Z' : \mathbb{P}(Z' \geq X_t) = 1$ $\forall t \in T$, $\mathbb{P}(Z' \geq Z) = 1$ (smallest probabilistic upper bound).

Lemma 3

Let $(X_t)_{t\in T}$ be any collection of rv.
An essential supremum always exists.
Furthermore, $\exists$ a countable $C \subset T$: $\sup_{t\in C} X_t = \operatorname{ess\,sup}_{t\in T} X_t$.

Let $U \sim U(0,1)$, $T = [0,1]$, and $X_t = 1_{\{c=t\}}$. $\sup_{t\in T} X_t = 1 \neq \operatorname{ess\,sup}_{t\in T} X_t = 0$.

Regularity from T Onward

Notation:

"$X \geq Y$" $\equiv \mathbb{P}(X \geq Y) = 1$.

"$X \geq Y$ on A" $\equiv \mathbb{P}(\{X \geq Y\} \cap A) = \mathbb{P}(A)$.

Regularity from T Onward

Notation:

"$X \geq Y$" $\equiv \mathbb{P}(X \geq Y) = 1$.

"$X \geq Y$ on A" $\equiv \mathbb{P}(\{X \geq Y\} \cap A) = \mathbb{P}(A)$.

Definition (Regularity from T onward)

$\tau \geq T$ is regular from T onward if for all $t \geq T$, $\mathbb{E}[Y_\tau \mid \mathcal{F}_t] > Y_t$ a.s. on $\{\tau > t\}$.

Regularity from T Onward

Notation:

"$X \geq Y$" $\equiv \mathbb{P}(X \geq Y) = 1$.

"$X \geq Y$ on A" $\equiv \mathbb{P}(\{X \geq Y\} \cap A) = \mathbb{P}(A)$.

Definition (Regularity from T onward)

$\tau \geq T$ is regular from T onward if for all $t \geq T$, $\mathbb{E}[Y_\tau \mid \mathcal{F}_t] > Y_t$ a.s. on $\{\tau > t\}$.

Lemma 1' (Regularity is wloo)

Under (A1), for any stopping time $\tau \geq T$ there exists a *regular* stopping time from T $\rho \geq T$ such that on $\rho \leq \tau$ with $\mathbb{E}[Y_\rho] \geq \mathbb{E}[Y_\tau]$.

Regularity from T Onward

Notation:

"$X \geq Y$" $\equiv \mathbb{P}(X \geq Y) = 1$.

"$X \geq Y$ on A" $\equiv \mathbb{P}(\{X \geq Y\} \cap A) = \mathbb{P}(A)$.

Definition (Regularity from T onward)

$\tau \geq T$ is regular from T onward if for all $t \geq T$, $\mathbb{E}[Y_\tau \mid \mathcal{F}_t] > Y_t$ a.s. on $\{\tau > t\}$.

Lemma 1' (Regularity is wloo)

Under (A1), for any stopping time $\tau \geq T$ there exists a *regular* stopping time from T $\rho \geq T$ such that on $\rho \leq \tau$ with $\mathbb{E}[Y_\rho] \geq \mathbb{E}[Y_\tau]$.

Lemma 2' (Regularity is closed under $\vee$)

Under (A1), if $\tau \geq T$ and $\rho \geq T$ are regular from T onward, then $\xi := \tau \vee \rho$ is regular from T onward and $\mathbb{E}[Y_\xi] \geq \max\{\mathbb{E}[Y_\tau], \mathbb{E}[Y_\rho]\}$.

Dynamic Programming Principle

Define:

$$V_t^* := \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}[Y_\tau \mid \mathcal{F}_t]$$

(optimise from t onward)

Dynamic Programming Principle

Define:

$$V_t^* := \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}[Y_\tau \mid \mathcal{F}_t]$$

(optimise from t onward)

Theorem (Dynamic Programming Principle)

Under (A1), $V_t^* = \max\{Y_t, \mathbb{E}[V_{t+1}^* \mid \mathcal{F}_t]\}$.

Dynamic Programming Principle

Define:

$$V_t^* := \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}[Y_\tau \mid \mathcal{F}_t]$$

(optimise from t onward)

Theorem (Dynamic Programming Principle)

Under (A1), $V_t^* = \max\{Y_t, \mathbb{E}[V_{t+1}^* \mid \mathcal{F}_t]\}$.

Proof

1. WTS $V_t^* \leq \max\{Y_t, \mathbb{E}[V_{t+1}^* \mid \mathcal{F}_t]\}$.

   Take any stopping time τ. On $\{\tau > t\}$, $\mathbb{E}[Y_\tau \mid \mathcal{F}_{t+1}] \leq V_{t+1}^*$.

Dynamic Programming Principle

Define:

$$V_t^* := \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}[Y_\tau \mid \mathcal{F}_t]$$

(optimise from t onward)

Theorem (Dynamic Programming Principle)

Under (A1), $V_t^* = \max\{Y_t, \mathbb{E}[V_{t+1}^* \mid \mathcal{F}_t]\}$.

Proof

1. WTS $V_t^* \leq \max\{Y_t, \mathbb{E}[V_{t+1}^* \mid \mathcal{F}_t]\}$.

   Take any stopping time τ. On $\{\tau > t\}$, $\mathbb{E}[Y_\tau \mid \mathcal{F}_{t+1}] \leq V_{t+1}^*$.

   $\implies \mathbb{E}[Y_\tau \mid \mathcal{F}_t] = \mathbb{E}[\mathbb{E}[Y_\tau \mid \mathcal{F}_{t+1}] \mid \mathcal{F}_t] \leq \mathbb{E}[V_{t+1}^* \mid \mathcal{F}_t]$.

Dynamic Programming Principle

Define:

$$V_t^* := \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}[Y_\tau \mid \mathcal{F}_t]$$

(optimise from t onward)

Theorem (Dynamic Programming Principle)

Under (A1), $V_t^* = \max\{Y_t, \mathbb{E}[V_{t+1}^* \mid \mathcal{F}_t]\}$.

Proof

1. WTS $V_t^* \leq \max\{Y_t, \mathbb{E}[V_{t+1}^* \mid \mathcal{F}_t]\}$.

   Take any stopping time τ. On $\{\tau > t\}$, $\mathbb{E}[Y_\tau \mid \mathcal{F}_{t+1}] \leq V_{t+1}^*$.

   $\implies \mathbb{E}[Y_\tau \mid \mathcal{F}_t] = \mathbb{E}[\mathbb{E}[Y_\tau \mid \mathcal{F}_{t+1}] \mid \mathcal{F}_t] \leq \mathbb{E}[V_{t+1}^* \mid \mathcal{F}_t]$.

   $\implies \mathbb{E}[Y_\tau \mid \mathcal{F}_t] = \mathbf{1}_{\{\tau=t\}} Y_t + \mathbf{1}_{\{\tau>t\}} \mathbb{E}[Y_\tau \mid \mathcal{F}_t] \leq \max\{Y_t, \mathbb{E}[V_{t+1}^* \mid \mathcal{F}_t]\}$.

Dynamic Programming Principle

Define:

$$V_t^* := \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}[Y_\tau \mid \mathcal{F}_t]$$

(optimise from t onward)

Theorem (Dynamic Programming Principle)

Under (A1), $V_t^* = \max\{Y_t, \mathbb{E}[V_{t+1}^* \mid \mathcal{F}_t]\}$.

Proof

1. WTS $V_t^* \leq \max\{Y_t, \mathbb{E}[V_{t+1}^* \mid \mathcal{F}_t]\}$.

   Take any stopping time τ. On $\{\tau > t\}$, $\mathbb{E}[Y_\tau \mid \mathcal{F}_{t+1}] \leq V_{t+1}^*$.

   $\implies \mathbb{E}[Y_\tau \mid \mathcal{F}_t] = \mathbb{E}[\mathbb{E}[Y_\tau \mid \mathcal{F}_{t+1}] \mid \mathcal{F}_t] \leq \mathbb{E}[V_{t+1}^* \mid \mathcal{F}_t]$.

   $\implies \mathbb{E}[Y_\tau \mid \mathcal{F}_t] = \mathbf{1}_{\{\tau=t\}} Y_t + \mathbf{1}_{\{\tau>t\}} \mathbb{E}[Y_\tau \mid \mathcal{F}_t] \leq \max\{Y_t, \mathbb{E}[V_{t+1}^* \mid \mathcal{F}_t]\}$.

   $\implies V_t^* \leq \max\{Y_t, \mathbb{E}[V_{t+1}^* \mid \mathcal{F}_t]\}$.

Dynamic Programming Principle

Define:

$$V_t^* := \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}[Y_\tau \mid \mathcal{F}_t]$$

(optimise from t onward)

Theorem (Dynamic Programming Principle)

Under (A1), $V_t^* = \max\{Y_t, \mathbb{E}[V_{t+1}^* \mid \mathcal{F}_t]\}$.

Proof

2. WTS $V_t^* \geq \max\{Y_t, \mathbb{E}[V_{t+1}^* \mid \mathcal{F}_t]\}$.

Dynamic Programming Principle

Define:

$$V_t^* := \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}[Y_\tau \mid \mathcal{F}_t]$$

(optimise from t onward)

Theorem (Dynamic Programming Principle)

Under (A1), $V_t^* = \max\{Y_t, \mathbb{E}[V_{t+1}^* \mid \mathcal{F}_t]\}$.

Proof

2. WTS $V_t^* \geq \max\{Y_t, \mathbb{E}[V_{t+1}^* \mid \mathcal{F}_t]\}$.

   By Lemma 3, $\exists$ sequence $\hat{\tau}_1, \hat{\tau}_2, \ldots$ with $\hat{\tau}_n \geq t+1$ such that $\mathbb{E}[Y_{\hat{\tau}_n} \mid \mathcal{F}_t] \to V_{t+1}^*$.

Dynamic Programming Principle

Define:

$$V_t^* := \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}[Y_\tau \mid \mathcal{F}_t]$$

(optimise from *t* onward)

Theorem (Dynamic Programming Principle)

Under (A1), $V_t^* = \max\{Y_t, \mathbb{E}[V_{t+1}^* \mid \mathcal{F}_t]\}$.

Proof

2. WTS $V_t^* \geq \max\{Y_t, \mathbb{E}[V_{t+1}^* \mid \mathcal{F}_t]\}$.

   By Lemma 3, $\exists$ sequence $\hat{\tau}_1, \hat{\tau}_2, \ldots$ with $\hat{\tau}_n \geq t+1$ such that $\mathbb{E}[Y_{\hat{\tau}_n} \mid \mathcal{F}_t] \to V_{t+1}^*$.

   Define regularised from $t+1$ onward $\rho_n := \inf\{\ell > t \mid \mathbb{E}[Y_{\hat{\tau}_n}] \leq Y_\ell\}$; let $\tau_n := \max\{\rho_1, \rho_2, \ldots, \rho_n\}$, regular from *t* onward.

Dynamic Programming Principle

Define:

$$V_t^* := \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}[Y_\tau \mid \mathcal{F}_t]$$

(optimise from t onward)

Theorem (Dynamic Programming Principle)

Under (A1), $V_t^* = \max\{Y_t, \mathbb{E}[V_{t+1}^* \mid \mathcal{F}_t]\}$.

Proof

2. WTS $V_t^* \geq \max\{Y_t, \mathbb{E}[V_{t+1}^* \mid \mathcal{F}_t]\}$.

   By Lemma 3, $\exists$ sequence $\hat{\tau}_1, \hat{\tau}_2, \ldots$ with $\hat{\tau}_n \geq t+1$ such that $\mathbb{E}[Y_{\hat{\tau}_n} \mid \mathcal{F}_t] \to V_{t+1}^*$.

   Define regularised from t+1 onward $\rho_n := \inf\{\ell > t \mid \mathbb{E}[Y_{\hat{\tau}_n}] \leq Y_\ell\}$; let $\tau_n := \max\{\rho_1, \rho_2, \ldots, \rho_n\}$, regular from t onward.

   By the lemmas 1' and 2',

$$V_t^* \geq \mathbb{E}[Y_{\tau_n} \mid \mathcal{F}_t] = \mathbb{E}[\mathbb{E}[Y_{\tau_n} \mid \mathcal{F}_{t+1}] \mid \mathcal{F}_t] \geq \mathbb{E}\left[\max_{\ell=1,\ldots,n} \mathbb{E}[Y_{\hat{\tau}_\ell} \mid \mathcal{F}_{t+1}] \middle| \mathcal{F}_t\right] \to \mathbb{E}[V_{t+1}^* \mid \mathcal{F}_t]$$

Dynamic Programming Principle

Define:

$$V_t^* := \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}[Y_\tau \mid \mathcal{F}_t]$$

(optimise from t onward)

Theorem (Dynamic Programming Principle)

Under (A1), $V_t^* = \max\{Y_t, \mathbb{E}[V_{t+1}^* \mid \mathcal{F}_t]\}$.

Proof

2. WTS $V_t^* \geq \max\{Y_t, \mathbb{E}[V_{t+1}^* \mid \mathcal{F}_t]\}$.

   By Lemma 3, $\exists$ sequence $\hat{\tau}_1, \hat{\tau}_2, \ldots$ with $\hat{\tau}_n \geq t+1$ such that $\mathbb{E}[Y_{\hat{\tau}_n} \mid \mathcal{F}_t] \to V_{t+1}^*$.

   Define regularised from $t+1$ onward $\rho_n := \inf\{\ell > t \mid \mathbb{E}[Y_{\hat{\tau}_n}] \leq Y_\ell\}$; let $\tau_n := \max\{\rho_1, \rho_2, \ldots, \rho_n\}$, regular from t onward.

   By the lemmas 1' and 2',

$$V_t^* \geq \mathbb{E}[Y_{\tau_n} \mid \mathcal{F}_t] = \mathbb{E}[\mathbb{E}[Y_{\tau_n} \mid \mathcal{F}_{t+1}] \mid \mathcal{F}_t] \geq \mathbb{E}\left[\max_{\ell=1,\ldots,n} \mathbb{E}[Y_{\hat{\tau}_\ell} \mid \mathcal{F}_{t+1}] \middle| \mathcal{F}_t\right] \to \mathbb{E}[V_{t+1}^* \mid \mathcal{F}_t]$$

   Since, trivially, $V_t^* \geq Y_t$, we get $V_t^* \geq \max\{Y_t, \mathbb{E}[V_{t+1}^* \mid \mathcal{F}_t]\}$.

Dynamic Programming Principle

$$V_t^* := \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}[Y_\tau \mid \mathcal{F}_t] = \max\{Y_t, \mathbb{E}[V_{t+1}^* \mid \mathcal{F}_t]\}; \qquad \tau^* := \inf\{t \geq 0 \mid Y_t = V_t^*\}$$

Example

Let $Y_0 := 0$, $Y_t := 1 - 1/t$ for $t \in \mathbb{N}$, $Y_\infty := 0$.

Satisfies (A1): $Y_t \leq 1$.

Fails (A2): $Y_t \to 1 > 0 = Y_\infty$.

Indeed, no optimal stopping time as $Y_t < Y_{t+1}$.

Note: $\tau^* = \infty$ and $Y_{\tau^*} = 0 < V_t^*$.

Characterising Optimal Stopping Time

$$V_t^* := \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}[Y_\tau \mid \mathcal{F}_t] = \max\{Y_t, \mathbb{E}[V_{t+1}^* \mid \mathcal{F}_t]\}; \qquad \tau^* := \inf\{t \geq 0 \mid Y_t = V_t^*\}$$

Lemma

Take any stopping time τ. Under (A1), $\mathbb{E}[Y_{\tau \wedge \tau^*}] \geq \mathbb{E}[Y_\tau]$.

Characterising Optimal Stopping Time

$$V_t^* := \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}[Y_\tau \mid \mathcal{F}_t] = \max\{Y_t, \mathbb{E}[V_{t+1}^* \mid \mathcal{F}_t]\}; \qquad \tau^* := \inf\{t \geq 0 \mid Y_t = V_t^*\}$$

Lemma

Take any stopping time τ. Under (A1), $\mathbb{E}[Y_{\tau \wedge \tau^*}] \geq \mathbb{E}[Y_\tau]$.

Stopping whenever τ^* says to stop can only improve the expected payoff.

Characterising Optimal Stopping Time

$$V_t^* := \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}[Y_\tau \mid \mathcal{F}_t] = \max\{Y_t, \mathbb{E}[V_{t+1}^* \mid \mathcal{F}_t]\}; \qquad \tau^* := \inf\{t \geq 0 \mid Y_t = V_t^*\}$$

Lemma

Take any stopping time τ. Under (A1), $\mathbb{E}[Y_{\tau \wedge \tau^*}] \geq \mathbb{E}[Y_\tau]$.

Proof

On $\{\tau^* = t < \tau\}$, $Y_{\tau^*} = Y_t = V_t^* \geq \mathbb{E}[Y_\tau \mid \mathcal{F}_t]$.

Characterising Optimal Stopping Time

$$V_t^* := \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}[Y_\tau \mid \mathcal{F}_t] = \max\{Y_t, \mathbb{E}[V_{t+1}^* \mid \mathcal{F}_t]\}; \qquad \tau^* := \inf\{t \geq 0 \mid Y_t = V_t^*\}$$

Lemma

Take any stopping time τ. Under (A1), $\mathbb{E}[Y_{\tau \wedge \tau^*}] \geq \mathbb{E}[Y_\tau]$.

Proof

On $\{\tau^* = t < \tau\}$, $Y_{\tau^*} = Y_t = V_t^* \geq \mathbb{E}[Y_\tau \mid \mathcal{F}_t]$. Hence,

$$\mathbb{E}[Y_{\tau \wedge \tau^*}] = \mathbb{E}[\mathbf{1}_{\{\tau^* < \tau\}} Y_{\tau^*}] + \mathbb{E}[\mathbf{1}_{\{\tau^* \geq \tau\}} Y_\tau]$$

Characterising Optimal Stopping Time

$$V_t^* := \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}[Y_\tau \mid \mathcal{F}_t] = \max\{Y_t, \mathbb{E}[V_{t+1}^* \mid \mathcal{F}_t]\}; \qquad \tau^* := \inf\{t \geq 0 \mid Y_t = V_t^*\}$$

Lemma

Take any stopping time τ. Under (A1), $\mathbb{E}[Y_{\tau \wedge \tau^*}] \geq \mathbb{E}[Y_\tau]$.

Proof

On $\{\tau^* = t < \tau\}$, $Y_{\tau^*} = Y_t = V_t^* \geq \mathbb{E}[Y_\tau \mid \mathcal{F}_t]$. Hence,

$$\begin{aligned}\mathbb{E}[Y_{\tau \wedge \tau^*}] &= \mathbb{E}[\mathbf{1}_{\{\tau^* < \tau\}} Y_{\tau^*}] + \mathbb{E}[\mathbf{1}_{\{\tau^* \geq \tau\}} Y_\tau] \\ &= \mathbb{E}\left[\sum_{t=0}^{\infty} \mathbf{1}_{\{\tau^* = t < \tau\}} Y_t\right] + \mathbb{E}[\mathbf{1}_{\{\tau^* \geq \tau\}} Y_\tau]\end{aligned}$$

Characterising Optimal Stopping Time

$$V_t^* := \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}[Y_\tau \mid \mathcal{F}_t] = \max\{Y_t, \mathbb{E}[V_{t+1}^* \mid \mathcal{F}_t]\}; \qquad \tau^* := \inf\{t \geq 0 \mid Y_t = V_t^*\}$$

Lemma

Take any stopping time τ. Under (A1), $\mathbb{E}[Y_{\tau \wedge \tau^*}] \geq \mathbb{E}[Y_\tau]$.

Proof

On $\{\tau^* = t < \tau\}$, $Y_{\tau^*} = Y_t = V_t^* \geq \mathbb{E}[Y_\tau \mid \mathcal{F}_t]$. Hence,

$$\begin{aligned}
\mathbb{E}[Y_{\tau \wedge \tau^*}] &= \mathbb{E}[\mathbf{1}_{\{\tau^* < \tau\}} Y_{\tau^*}] + \mathbb{E}[\mathbf{1}_{\{\tau^* \geq \tau\}} Y_\tau] \\
&= \mathbb{E}\left[\sum_{t=0}^{\infty} \mathbf{1}_{\{\tau^* = t < \tau\}} Y_t\right] + \mathbb{E}[\mathbf{1}_{\{\tau^* \geq \tau\}} Y_\tau] \\
&= \mathbb{E}\left[\sum_{t=0}^{\infty} \mathbf{1}_{\{\tau^* = t < \tau\}} V_t^*\right] + \mathbb{E}[\mathbf{1}_{\{\tau^* \geq \tau\}} Y_\tau]
\end{aligned}$$

Characterising Optimal Stopping Time

$$V_t^* := \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}[Y_\tau \mid \mathcal{F}_t] = \max\{Y_t, \mathbb{E}[V_{t+1}^* \mid \mathcal{F}_t]\}; \qquad \tau^* := \inf\{t \geq 0 \mid Y_t = V_t^*\}$$

Lemma

Take any stopping time τ. Under (A1), $\mathbb{E}[Y_{\tau \wedge \tau^*}] \geq \mathbb{E}[Y_\tau]$.

Proof

On $\{\tau^* = t < \tau\}$, $Y_{\tau^*} = Y_t = V_t^* \geq \mathbb{E}[Y_\tau \mid \mathcal{F}_t]$. Hence,

$$\begin{aligned}
\mathbb{E}[Y_{\tau \wedge \tau^*}] &= \mathbb{E}[\mathbf{1}_{\{\tau^* < \tau\}} Y_{\tau^*}] + \mathbb{E}[\mathbf{1}_{\{\tau^* \geq \tau\}} Y_\tau] \\
&= \mathbb{E}\left[\sum_{t=0}^{\infty} \mathbf{1}_{\{\tau^* = t < \tau\}} Y_t\right] + \mathbb{E}[\mathbf{1}_{\{\tau^* \geq \tau\}} Y_\tau] \\
&= \mathbb{E}\left[\sum_{t=0}^{\infty} \mathbf{1}_{\{\tau^* = t < \tau\}} V_t^*\right] + \mathbb{E}[\mathbf{1}_{\{\tau^* \geq \tau\}} Y_\tau] \\
&\geq \mathbb{E}\left[\sum_{t=0}^{\infty} \mathbf{1}_{\{\tau^* = t < \tau\}} \mathbb{E}[Y_\tau \mid \mathcal{F}_t]\right] + \mathbb{E}[\mathbf{1}_{\{\tau^* \geq \tau\}} Y_\tau] = \mathbb{E}[Y_\tau].
\end{aligned}$$

Characterising Optimal Stopping Time

$$V_t^* := \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}[Y_\tau \mid \mathcal{F}_t] = \max\{Y_t, \mathbb{E}[V_{t+1}^* \mid \mathcal{F}_t]\}; \qquad \tau^* := \inf\{t \geq 0 \mid Y_t = V_t^*\}$$

Theorem (Optimal Stopping Time)

Under (A1), if an optimal stopping time exists, τ^* is optimal.

Characterising Optimal Stopping Time

$$V_t^* := \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}[Y_\tau \mid \mathcal{F}_t] = \max\{Y_t, \mathbb{E}[V_{t+1}^* \mid \mathcal{F}_t]\}; \qquad \tau^* := \inf\{t \geq 0 \mid Y_t = V_t^*\}$$

Theorem (Optimal Stopping Time)

Under (A1), if an optimal stopping time exists, τ^* is optimal.

Proof

Let τ be an optimal stopping time.

Characterising Optimal Stopping Time

$$V_t^* := \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}[Y_\tau \mid \mathcal{F}_t] = \max\{Y_t, \mathbb{E}[V_{t+1}^* \mid \mathcal{F}_t]\}; \qquad \tau^* := \inf\{t \geq 0 \mid Y_t = V_t^*\}$$

Theorem (Optimal Stopping Time)

Under (A1), if an optimal stopping time exists, τ^* is optimal.

Proof

Let τ be an optimal stopping time.

By Lemma 4, $\tau' := \tau \wedge \tau^*$ must also be optimal.

Characterising Optimal Stopping Time

$$V_t^* := \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}[Y_\tau \mid \mathcal{F}_t] = \max\{Y_t, \mathbb{E}[V_{t+1}^* \mid \mathcal{F}_t]\}; \qquad \tau^* := \inf\{t \geq 0 \mid Y_t = V_t^*\}$$

Theorem (Optimal Stopping Time)

Under (A1), if an optimal stopping time exists, τ^* is optimal.

Proof

Let τ be an optimal stopping time.

By Lemma 4, $\tau' := \tau \wedge \tau^*$ must also be optimal.

By Lemma 1, there is a regular τ'' : $\mathbb{E}[Y_{\tau''}] \geq \mathbb{E}[Y_{\tau'}]$ and $\tau'' \leq \tau' \leq \tau^*$. Hence, τ'' must also be optimal.

Characterising Optimal Stopping Time

$$V_t^* := \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}[Y_\tau \mid \mathcal{F}_t] = \max\{Y_t, \mathbb{E}[V_{t+1}^* \mid \mathcal{F}_t]\}; \qquad \tau^* := \inf\{t \geq 0 \mid Y_t = V_t^*\}$$

Theorem (Optimal Stopping Time)

Under (A1), if an optimal stopping time exists, τ^* is optimal.

Proof

Let τ be an optimal stopping time.

By Lemma 4, $\tau' := \tau \wedge \tau^*$ must also be optimal.

By Lemma 1, there is a regular τ'' : $\mathbb{E}[Y_{\tau''}] \geq \mathbb{E}[Y_{\tau'}]$ and $\tau'' \leq \tau' \leq \tau^*$. Hence, τ'' must also be optimal.

Finally, by Lemma 2, $\tau'' \vee \tau^*$ must also be optimal. Note that $\tau'' \vee \tau^* = \tau^*$ by construction.

Characterising Optimal Stopping Time

$$V_t^* := \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}[Y_\tau \mid \mathcal{F}_t] = \max\{Y_t, \mathbb{E}[V_{t+1}^* \mid \mathcal{F}_t]\};$$

$$\tau^* := \inf\{t \geq 0 \mid Y_t = V_t^*\} = \inf\{t \geq 0 \mid Y_t \geq \mathbb{E}[V_{t+1}^* \mid \mathcal{F}_t]\}$$

It can be shown that τ^* is the **earliest optimal stopping time**, i.e., $\tau^* \leq \tau \; \forall$ optimal τ.
(Intuition: If $\tau = t < \tau^*$, then $Y_t < V_t^*$ and an improvement can be reached)

Characterising Optimal Stopping Time

$$V_t^* := \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}[Y_\tau \mid \mathcal{F}_t] = \max\{Y_t, \mathbb{E}[V_{t+1}^* \mid \mathcal{F}_t]\};$$

$$\tau^* := \inf\{t \geq 0 \mid Y_t = V_t^*\} = \inf\{t \geq 0 \mid Y_t \geq \mathbb{E}[V_{t+1}^* \mid \mathcal{F}_t]\}$$

It can be shown that τ^* is the **earliest optimal stopping time**, i.e., $\tau^* \leq \tau \; \forall$ optimal τ.
(Intuition: If $\tau = t < \tau^*$, then $Y_t < V_t^*$ and an improvement can be reached)

Another stopping time: $\tau^{**} := \inf\{t \geq 0 \mid Y_t > \mathbb{E}[V_{t+1}^* \mid \mathcal{F}_t]\}$

It can be shown that τ^{**} is the **latest optimal stopping time**, i.e., $\tau \leq \tau^{**} \; \forall$ optimal τ.

Overview

Satisticing

Oftentimes DM don’t consider all items (virtually impossible in online shopping...).

DM knows there is a large set of feasible items but doesn’t quite known what they are.
  Upon stopping their search, pick best item available.

When to stop searching?

Satisticing

Oftentimes DM don't consider all items (virtually impossible in online shopping...).

DM knows there is a large set of feasible items but doesn't quite known what they are. Upon stopping their search, pick best item available.

When to stop searching?

Setup

DM faces a large choice set A with T items.

Parsing through the item list bears a cost $c > 0$.

Prior about the value of each option $X_t \overset{iid}{\sim} F$, absolutely continuous, strictly increasing.

Satisticing

Oftentimes DM don't consider all items (virtually impossible in online shopping...).

DM knows there is a large set of feasible items but doesn't quite known what they are. Upon stopping their search, pick best item available.

When to stop searching?

Setup

- DM faces a large choice set A with T items.
- Parsing through the item list bears a cost $c > 0$.
- Prior about the value of each option $X_t \overset{iid}{\sim} F$, absolutely continuous, strictly increasing.

Proposition

Let $M_t := \max_{s \leq t} X_t$ and $\bar{x} : c = \int_{\bar{x}}^{\infty} (X - \bar{x})\, \mathrm{d}F(X)$. Then, $\tau_T^* := \inf\{t \geq 0 \mid M_t \geq \bar{x}\} \wedge T$ is optimal.

Satisticing

Solving the Problem (Backwards induction intuition)

At $T-1$: stop and get $M_{T-1} - (T-1)c$ or continue and get $\mathbb{E}[M_T \mid \mathcal{F}_{T-1}] - Tc$.

Satisticing

Solving the Problem (Backwards induction intuition)

At $T-1$**:** stop and get $M_{T-1} - (T-1)c$ or continue and get $\mathbb{E}[M_T \mid \mathcal{F}_{T-1}] - Tc$.

$$M_{T-1} - (T-1)c \leq \mathbb{E}[M_T \mid \mathcal{F}_{T-1}] - Tc = \int_{-\infty}^{M_{T-1}} M_{T-1}\,\mathrm{d}F(X) + \int_{M_{T-1}}^{\infty} X\,\mathrm{d}F(X) - Tc \iff$$

$$c \leq \int_{M_{T-1}}^{\infty} (x - M_{T-1})\,\mathrm{d}F(X). \qquad \bar{x} : c = \int_{\bar{x}}^{\infty} (X - \bar{x})\,\mathrm{d}F(X).$$

Satisticing

Solving the Problem (Backwards induction intuition)

At $T-1$: stop and get $M_{T-1} - (T-1)c$ or continue and get $\mathbb{E}[M_T \mid \mathcal{F}_{T-1}] - Tc$.

$$M_{T-1} - (T-1)c \leq \mathbb{E}[M_T \mid \mathcal{F}_{T-1}] - Tc = \int_{-\infty}^{M_{T-1}} M_{T-1}\,\mathrm{d}F(X) + \int_{M_{T-1}}^{\infty} X\,\mathrm{d}F(X) - Tc \iff$$
$$c \leq \int_{M_{T-1}}^{\infty} (x - M_{T-1})\,\mathrm{d}F(X). \qquad \bar{x} : c = \int_{\bar{x}}^{\infty} (X - \bar{x})\,\mathrm{d}F(X).$$

Stop at $T-1$ if $M_{T-1} \geq \bar{x}$.

Satisticing

Solving the Problem (Backwards induction intuition)

At $T-\mathbf{1}$**:** stop and get $M_{T-1} - (T-1)c$ or continue and get $\mathbb{E}[M_T \mid \mathcal{F}_{T-1}] - Tc$.

$$M_{T-1} - (T-1)c \leq \mathbb{E}[M_T \mid \mathcal{F}_{T-1}] - Tc = \int_{-\infty}^{M_{T-1}} M_{T-1}\,\mathrm{d}F(X) + \int_{M_{T-1}}^{\infty} X\,\mathrm{d}F(X) - Tc \iff$$

$$c \leq \int_{M_{T-1}}^{\infty} (x - M_{T-1})\,\mathrm{d}F(X). \qquad \bar{x} : c = \int_{\bar{x}}^{\infty} (X - \bar{x})\,\mathrm{d}F(X).$$

Stop at $T-1$ if $M_{T-1} \geq \bar{x}$.

At $T-\mathbf{2}$**:** stop and get $M_{T-2} - (T-2)c$ or continue and get $\mathbb{E}[M_{T-1} \mid \mathcal{F}_{T-2}] - (T-1)c$.

Satisticing

Solving the Problem (Backwards induction intuition)

At $T-1$: stop and get $M_{T-1}-(T-1)c$ or continue and get $\mathbb{E}[M_T \mid \mathcal{F}_{T-1}] - Tc$.

$M_{T-1}-(T-1)c \leq \mathbb{E}[M_T \mid \mathcal{F}_{T-1}] - Tc = \int_{-\infty}^{M_{T-1}} M_{T-1}\,\mathrm{d}F(X) + \int_{M_{T-1}}^{\infty} X\,\mathrm{d}F(X) - Tc \iff$
$c \leq \int_{M_{T-1}}^{\infty}(x - M_{T-1})\,\mathrm{d}F(X)$. $\quad \bar{x} : c = \int_{\bar{x}}^{\infty}(X-\bar{x})\,\mathrm{d}F(X)$.

Stop at $T-1$ if $M_{T-1} \geq \bar{x}$.

At $T-2$: stop and get $M_{T-2}-(T-2)c$ or continue and get $\mathbb{E}[M_{T-1} \mid \mathcal{F}_{T-2}] - (T-1)c$.

Suppose $M_{T-2} \geq \bar{x}$. Then upon continuing would stop at $T-1$ and get $\max\{M_{T-2}, X_{T-1}\} - (T-1)c$.

Satisticing

Solving the Problem (Backwards induction intuition)

At $T-1$: stop and get $M_{T-1} - (T-1)c$ or continue and get $\mathbb{E}[M_T \mid \mathcal{F}_{T-1}] - Tc$.

$M_{T-1} - (T-1)c \leq \mathbb{E}[M_T \mid \mathcal{F}_{T-1}] - Tc = \int_{-\infty}^{M_{T-1}} M_{T-1} \, \mathrm{d}F(X) + \int_{M_{T-1}}^{\infty} X \, \mathrm{d}F(X) - Tc \iff$
$c \leq \int_{M_{T-1}}^{\infty} (x - M_{T-1}) \, \mathrm{d}F(X)$. $\quad \bar{x} : c = \int_{\bar{x}}^{\infty} (X - \bar{x}) \, \mathrm{d}F(X)$.

Stop at $T-1$ if $M_{T-1} \geq \bar{x}$.

At $T-2$: stop and get $M_{T-2} - (T-2)c$ or continue and get $\mathbb{E}[M_{T-1} \mid \mathcal{F}_{T-2}] - (T-1)c$.

Suppose $M_{T-2} \geq \bar{x}$. Then upon continuing would stop at $T-1$ and get $\max\{M_{T-2}, X_{T-1}\} - (T-1)c$.

Better to stop now and get $M_{T-2} - (T-2)c$ if

$$M_{T-2} - (T-2)c \leq \mathbb{E}[M_{T-1} \mid \mathcal{F}_{T-2}] - (T-1)c \iff c \leq \int_{M_{T-2}}^{\infty} (X - M_{T-2}) \, \mathrm{d}F(X)$$

$$\iff M_{T-2} \geq \bar{x} \qquad \text{(as is by assumption).}$$

Satisticing

Solving the Problem (Backwards induction intuition)

At $T-1$: stop and get $M_{T-1} - (T-1)c$ or continue and get $\mathbb{E}[M_T \mid \mathcal{F}_{T-1}] - Tc$.

$M_{T-1} - (T-1)c \leq \mathbb{E}[M_T \mid \mathcal{F}_{T-1}] - Tc = \int_{-\infty}^{M_{T-1}} M_{T-1}\, \mathrm{d}F(X) + \int_{M_{T-1}}^{\infty} X\, \mathrm{d}F(X) - Tc \iff$
$c \leq \int_{M_{T-1}}^{\infty} (x - M_{T-1})\, \mathrm{d}F(X)$. $\quad \bar{x} : c = \int_{\bar{x}}^{\infty} (X - \bar{x})\, \mathrm{d}F(X)$.

Stop at $T-1$ if $M_{T-1} \geq \bar{x}$.

At $T-2$: stop and get $M_{T-2} - (T-2)c$ or continue and get $\mathbb{E}[M_{T-1} \mid \mathcal{F}_{T-2}] - (T-1)c$.

Suppose $M_{T-2} \geq \bar{x}$. Then upon continuing would stop at $T-1$ and get $\max\{M_{T-2}, X_{T-1}\} - (T-1)c$.

Better to stop now and get $M_{T-2} - (T-2)c$ if

$$M_{T-2} - (T-2)c \leq \mathbb{E}[M_{T-1} \mid \mathcal{F}_{T-2}] - (T-1)c \iff c \leq \int_{M_{T-2}}^{\infty} (X - M_{T-2})\, \mathrm{d}F(X)$$

$\iff M_{T-2} \geq \bar{x}$ (as is by assumption).

Suppose $M_{T-2} < \bar{x}$. Then, if it were to end at $T-1$ would anyway continue; more so given the option value.

Satisticing

Solving the Problem (Backwards induction intuition)

At $T-1$**:** stop and get $M_{T-1}-(T-1)c$ or continue and get $\mathbb{E}[M_T \mid \mathcal{F}_{T-1}]-Tc$.

$M_{T-1}-(T-1)c \leq \mathbb{E}[M_T \mid \mathcal{F}_{T-1}]-Tc = \int_{-\infty}^{M_{T-1}} M_{T-1}\,\mathrm{d}F(X) + \int_{M_{T-1}}^{\infty} X\,\mathrm{d}F(X) - Tc \iff$
$c \leq \int_{M_{T-1}}^{\infty}(x-M_{T-1})\,\mathrm{d}F(X)$. $\quad \bar{x} : c = \int_{\bar{x}}^{\infty}(X-\bar{x})\,\mathrm{d}F(X)$.

Stop at $T-1$ if $M_{T-1} \geq \bar{x}$.

At $T-2$**:** stop and get $M_{T-2}-(T-2)c$ or continue and get $\mathbb{E}[M_{T-1} \mid \mathcal{F}_{T-2}]-(T-1)c$.

Suppose $M_{T-2} \geq \bar{x}$. Then upon continuing would stop at $T-1$ and get $\max\{M_{T-2}, X_{T-1}\}-(T-1)c$.

Better to stop now and get $M_{T-2}-(T-2)c$ if

$$M_{T-2}-(T-2)c \leq \mathbb{E}[M_{T-1} \mid \mathcal{F}_{T-2}]-(T-1)c \iff c \leq \int_{M_{T-2}}^{\infty}(X-M_{T-2})\,\mathrm{d}F(X)$$

$$\iff M_{T-2} \geq \bar{x} \qquad \text{(as is by assumption).}$$

Suppose $M_{T-2} < \bar{x}$. Then, if it were to end at $T-1$ would anyway continue; more so given the option value.

Conclusion: Stop at $T-2$ if $M_{T-2} \geq \bar{x}$.

Satisticing

Solving the Problem (Backwards induction intuition)

At $T-1$**:** stop and get $M_{T-1} - (T-1)c$ or continue and get $\mathbb{E}[M_T \mid \mathcal{F}_{T-1}] - Tc$.

$M_{T-1} - (T-1)c \leq \mathbb{E}[M_T \mid \mathcal{F}_{T-1}] - Tc = \int_{-\infty}^{M_{T-1}} M_{T-1}\,\mathrm{d}F(X) + \int_{M_{T-1}}^{\infty} X\,\mathrm{d}F(X) - Tc \iff$
$c \leq \int_{M_{T-1}}^{\infty} (x - M_{T-1})\,\mathrm{d}F(X)$. $\quad \bar{x} : c = \int_{\bar{x}}^{\infty} (X - \bar{x})\,\mathrm{d}F(X)$.

Stop at $T-1$ if $M_{T-1} \geq \bar{x}$.

At $T-2$**:** stop and get $M_{T-2} - (T-2)c$ or continue and get $\mathbb{E}[M_{T-1} \mid \mathcal{F}_{T-2}] - (T-1)c$.

Suppose $M_{T-2} \geq \bar{x}$. Then upon continuing would stop at $T-1$ and get $\max\{M_{T-2}, X_{T-1}\} - (T-1)c$.

Better to stop now and get $M_{T-2} - (T-2)c$ if

$$M_{T-2} - (T-2)c \leq \mathbb{E}[M_{T-1} \mid \mathcal{F}_{T-2}] - (T-1)c \iff c \leq \int_{M_{T-2}}^{\infty} (X - M_{T-2})\,\mathrm{d}F(X)$$

$$\iff M_{T-2} \geq \bar{x} \qquad \text{(as is by assumption).}$$

Suppose $M_{T-2} < \bar{x}$. Then, if it were to end at $T-1$ would anyway continue; more so given the option value.

Conclusion: Stop at $T-2$ if $M_{T-2} \geq \bar{x}$.

Induction: $\tau_T^* := \inf\{t \geq 0 \mid M_t \geq \bar{x}\} \wedge T$ is optimal. (handwavy; rigorous proof later.)

Satisticing

Satisficing solution: DM stops whenever has seen something "good enough"

$\tau_T^* = \inf\{t \geq \mathbf{0} \mid M_t \geq \bar{x}\} \wedge T.$

Remark

$\tau_T^* = t < T \implies M_{\tau_T^*} = X_t.$

Stop only if last item seen is best! $\tau_T^* = t < T \implies M_{\tau_T^*} = X_t.$

Satisticing

Satisficing solution: DM stops whenever has seen something “good enough”

$\tau_T^* = \inf\{t \geq 0 \mid M_t \geq \bar{x}\} \wedge T$.

Remark

$\tau_T^* = t < T \implies M_{\tau_T^*} = X_t$.

Stop only if last item seen is best! $\tau_T^* = t < T \implies M_{\tau_T^*} = X_t$.

Proposition

DM chooses X_t if $X_t \wedge \bar{x} > \max_{s \neq t} X_s \wedge \bar{x}$ and only if $X_t \wedge \bar{x} \geq \max_{s \neq t} X_s \wedge \bar{x}$.

Proposition

DM chooses X_t if $X_t \wedge \bar{x} > \max_{s \neq t} X_s \wedge \bar{x}$ and only if $X_t \wedge \bar{x} \geq \max_{s \neq t} X_s \wedge \bar{x}$.

Proof

If part.

Suppose $X_t \wedge \bar{x} > \max_{s \neq t} X_s \wedge \bar{x}$.

Satisticing

Proposition

DM chooses X_t if $X_t \wedge \bar{x} > \max_{s \neq t} X_s \wedge \bar{x}$ and only if $X_t \wedge \bar{x} \geq \max_{s \neq t} X_s \wedge \bar{x}$.

Proof

If part.

Suppose $X_t \wedge \bar{x} > \max_{s \neq t} X_s \wedge \bar{x}$.

Case 1. $X_t \geq \bar{x}$.

Satisticing

Proposition

DM chooses X_t if $X_t \wedge \bar{x} > \max_{s \neq t} X_s \wedge \bar{x}$ and only if $X_t \wedge \bar{x} \geq \max_{s \neq t} X_s \wedge \bar{x}$.

Proof

If part.

Suppose $X_t \wedge \bar{x} > \max_{s \neq t} X_s \wedge \bar{x}$.

Case 1. $X_t \geq \bar{x}$.

Then, $\bar{x} > \max_{s \neq t} X_s$ and DM won't stop before t because $M_{t-1} \leq \max_{s \neq t} X_s < \bar{x}$.

Satisticing

Proposition

DM chooses X_t if $X_t \wedge \bar{x} > \max_{s \neq t} X_s \wedge \bar{x}$ and only if $X_t \wedge \bar{x} \geq \max_{s \neq t} X_s \wedge \bar{x}$.

Proof

If part.

Suppose $X_t \wedge \bar{x} > \max_{s \neq t} X_s \wedge \bar{x}$.

Case 1. $X_t \geq \bar{x}$.

Then, $\bar{x} > \max_{s \neq t} X_s$ and DM won't stop before t because $M_{t-1} \leq \max_{s \neq t} X_s < \bar{x}$.

At t, the DM stops ($X_t \geq \bar{x}$) and chooses t ($X_t > M_{t-1}$).

Satisticing

Proposition

DM chooses X_t if $X_t \wedge \bar{x} > \max_{s \neq t} X_s \wedge \bar{x}$ and only if $X_t \wedge \bar{x} \geq \max_{s \neq t} X_s \wedge \bar{x}$.

Proof

If part.

Suppose $X_t \wedge \bar{x} > \max_{s \neq t} X_s \wedge \bar{x}$.

Case 1. $X_t \geq \bar{x}$.

Then, $\bar{x} > \max_{s \neq t} X_s$ and DM won't stop before t because $M_{t-1} \leq \max_{s \neq t} X_s < \bar{x}$.

At t, the DM stops ($X_t \geq \bar{x}$) and chooses t ($X_t > M_{t-1}$).

Case 2. $\bar{x} > X_t > \max_{s \neq t} X_s \wedge \bar{x}$. Never stop until T; after going through all the items, choose $X_t > \max_{s \neq t} X_s$.

Satisticing

Proposition

DM chooses X_t if $X_t \wedge \bar{x} > \max_{s \neq t} X_s \wedge \bar{x}$ and only if $X_t \wedge \bar{x} \geq \max_{s \neq t} X_s \wedge \bar{x}$.

Proof

If part.

Suppose $X_t \wedge \bar{x} > \max_{s \neq t} X_s \wedge \bar{x}$.

Case 1. $X_t \geq \bar{x}$.

Then, $\bar{x} > \max_{s \neq t} X_s$ and DM won't stop before t because $M_{t-1} \leq \max_{s \neq t} X_s < \bar{x}$.

At t, the DM stops ($X_t \geq \bar{x}$) and chooses t ($X_t > M_{t-1}$).

Case 2. $\bar{x} > X_t > \max_{s \neq t} X_s \wedge \bar{x}$. Never stop until T; after going through all the items, choose $X_t > \max_{s \neq t} X_s$.

Only if part.

Suppose $X_t \wedge \bar{x} < \max_{s \neq t} X_s \wedge \bar{x}$.

Satisticing

Proposition

DM chooses X_t if $X_t \wedge \bar{x} > \max_{s \neq t} X_s \wedge \bar{x}$ and only if $X_t \wedge \bar{x} \geq \max_{s \neq t} X_s \wedge \bar{x}$.

Proof

If part.

Suppose $X_t \wedge \bar{x} > \max_{s \neq t} X_s \wedge \bar{x}$.

Case 1. $X_t \geq \bar{x}$.

Then, $\bar{x} > \max_{s \neq t} X_s$ and DM won't stop before t because $M_{t-1} \leq \max_{s \neq t} X_s < \bar{x}$.

At t, the DM stops ($X_t \geq \bar{x}$) and chooses t ($X_t > M_{t-1}$).

Case 2. $\bar{x} > X_t > \max_{s \neq t} X_s \wedge \bar{x}$. Never stop until T; after going through all the items, choose $X_t > \max_{s \neq t} X_s$.

Only if part.

Suppose $X_t \wedge \bar{x} < \max_{s \neq t} X_s \wedge \bar{x}$. Then X_t is never chosen.

Satisticing

Satisficing solution: DM stops whenever has seen something “good enough”

$\tau_T^* = \inf\{t \geq \mathbf{0} \mid M_t \geq \bar{x}\} \wedge T.$

Remark

$\tau_T^* = t < T \implies M_{\tau_T^*} = X_t.$

Proposition

DM chooses X_t if $X_t \wedge \bar{x} > \max_{s \neq t} X_s \wedge \bar{x}$ and only if $X_t \wedge \bar{x} \geq \max_{s \neq t} X_s \wedge \bar{x}$.

Corollary

$\mathbb{E}[X_{\tau_T^*}] = \mathbb{E}[\max_{t \leq T} X_t \wedge \bar{x}].$

Dependence on c only through $\bar{x}$.

Satisticing

Remark

$\mathbb{E}[\tau_T^*] = \frac{1-F(\bar{x})^{T-1}}{1-F(\bar{x})}$.

Proof

Since $\mathbb{P}(\tau_T^* \geq t) = 1-\mathbb{P}(\tau_T^* \leq t-1) = 1-1_{\{t\leq T\}}\sum_{s=1}^{t-1}(1-F(\bar{x}))F(\bar{x})^{s-1} = 1-1_{\{t\leq T\}}(1-F(\bar{x})^{t-1})$.

Then, $\mathbb{E}[\tau_T^*] = \sum_{t=1}^{T}\mathbb{P}(\tau_T^* \geq t) = \frac{1-F(\bar{x})^{T-1}}{1-F(\bar{x})}$.

Note that

$\text{sign}(\frac{\partial}{\partial \bar{x}}\mathbb{E}[\tau_T^*]) = \text{sign}(F(\bar{x})+F(\bar{x})^{T+1}(T-1)-F(\bar{x})^{T}T) = \text{sign}(1+F(\bar{x})^{T}(T-1)-F(\bar{x})^{T-1}T) > 0$ for $F(\bar{x}) \in (0,1)$.

Comparative Statics

Remark

(i) $\uparrow c \implies \downarrow \bar{x} \implies \downarrow \mathbb{E}[X_{\tau_T^*}], \mathbb{E}[\tau_T^*]$;

(ii) F' MPS of $F \implies \bar{x}' \geq \bar{x}$ (higher option value) $\implies \uparrow \mathbb{E}[X_{\tau_T^*}], \mathbb{E}[\tau_T^*]$;

(iii) $F'(x) = F(x - \mu)$ (shift in mean) $\implies \bar{x}' = \bar{x} + \mu$
$\implies \mathbb{E}[X'_{\tau_T^*}] = \mathbb{E}[X_{\tau_T^*}] + \mu, \quad \mathbb{E}[\tau_T^*] = \mathbb{E}[\tau_T^{*\prime}]$;

(iv) $\bar{x}$ remains constant wrt $T \implies$ so does $\mathbb{E}[X_{\tau_T^*}], \mathbb{E}[\tau_T^*]$.

Overview

Setup and Assumptions

$\{X_0, X_1, X_2, ...\}$ rv whose joint distribution is assumed to be known; write $X^t := (X_\ell)_{\ell=1,...,t}$.

Sequence of functions $x^t \mapsto y_t(x^t) \in \mathbb{R}$; write $Y_t := y_t(x^t)$.

Filtration $\mathbb{F} = \{\mathcal{F}_t\} = \sigma(X^t)$.

Adapted payoff process $\{Y_t\}$; terminal Y_∞ (possibly $-\infty$).

Stopping time τ: $\{\tau \leq t\} \in \mathcal{F}_t$ for all t; feasible set $\mathbb{T}$.

Truncation: T terminal time; Stopping times: $\mathbb{T}_T = \{\tau \leq T\}$.

Objective: maximise value of Y by adequately choosing stopping time,

$$V_0^{(T)} := \sup_{\tau \in \mathbb{T}_T} \mathbb{E}[Y_\tau].$$

Note: T finite $\not\Longrightarrow$ $\mathbb{T}_T$ finite (not finite in general)

Setup and Assumptions

$\{X_0, X_1, X_2, ...\}$ rv whose joint distribution is assumed to be known; write $X^t := (X_\ell)_{\ell=1,...,t}$.

Sequence of functions $x^t \mapsto y_t(x^t) \in \mathbb{R}$; write $Y_t := y_t(x^t)$.

Filtration $\mathbb{F} = \{\mathcal{F}_t\} = \sigma(X^t)$.

Adapted payoff process $\{Y_t\}$; terminal Y_∞ (possibly $-\infty$).

Stopping time τ: $\{\tau \leq t\} \in \mathcal{F}_t$ for all t; feasible set $\mathbb{T}$.

Truncation: T terminal time; Stopping times: $\mathbb{T}_T = \{\tau \leq T\}$.

Objective: maximise value of Y by adequately choosing stopping time,

$$V_0^{(T)} := \sup_{\tau \in \mathbb{T}_T} \mathbb{E}[Y_\tau].$$

Note: T finite $\not\Longleftarrow$ $\mathbb{T}_T$ finite (not finite in general)

Backward Induction: $V_T^{(T)} := Y_T$; $V_t^{(T)} := \max\{Y_t, \mathbb{E}[V_{t+1}^{(T)} \mid \mathcal{F}_t]\}$, $t = T-1, T-2, ..., 0$.

Guarantees existence of optimal stopping time.

Setup and Assumptions

$\{X_0, X_1, X_2, \ldots\}$ rv whose joint distribution is assumed to be known; write $X^t := (X_\ell)_{\ell=1,\ldots,t}$.

Sequence of functions $x^t \mapsto y_t(x^t) \in \mathbb{R}$; write $Y_t := y_t(x^t)$.

Filtration $\mathbb{F} = \{\mathcal{F}_t\} = \sigma(X^t)$.

Adapted payoff process $\{Y_t\}$; terminal Y_∞ (possibly $-\infty$).

Stopping time τ: $\{\tau \leq t\} \in \mathcal{F}_t$ for all t; feasible set $\mathbb{T}$.

Truncation: T terminal time; Stopping times: $\mathbb{T}_T = \{\tau \leq T\}$.

Objective: maximise value of Y by adequately choosing stopping time,

$$V_0^{(T)} := \sup_{\tau \in \mathbb{T}_T} \mathbb{E}[Y_\tau].$$

Note: T finite $\not\Longleftarrow$ $\mathbb{T}_T$ finite (not finite in general)

Backward Induction: $V_T^{(T)} := Y_T$; $V_t^{(T)} := \max\{Y_t, \mathbb{E}[V_{t+1}^{(T)} \mid \mathcal{F}_t]\}$, $t = T-1, T-2, \ldots, 0$.

Guarantees existence of optimal stopping time.

For truncation in problems when continuing forever is valuable, replace

$$Y_T^{(T)} := \max\{Y_T, \mathbb{E}[Y_\infty \mid \mathcal{F}_T]\}$$

Simple Stopping Rules

One-Stage Look-Ahead Stopping Time: $\tau_{\text{1-sla}} := \inf\{t \geq 0 \mid Y_t \geq \mathbb{E}[Y_{t+1} \mid \mathcal{F}_t]\}$.

Myopic stopping: stop as soon as it doesn't pay off continuing for one more period.

Simple Stopping Rules

One-Stage Look-Ahead Stopping Time: $\tau_{1\text{-sla}} := \inf\{t \geq 0 \mid Y_t \geq \mathbb{E}[Y_{t+1} \mid \mathcal{F}_t]\}$.

Myopic stopping: stop as soon as it doesn't pay off continuing for one more period.

***k*-Stage Look-Ahead Stopping Time:** $\tau_{k\text{-sla}} := \inf\{t \geq 0 \mid Y_t \geq \mathbb{E}[V_{t+1}^{(t+k)} \mid \mathcal{F}_t]\}$.

Stop if continuing for at most k more periods isn't worthwhile.

Naive: not considering that next period will consider the following k periods instead of $k-1$.

Note: $Y_t \geq \mathbb{E}[V_{t+1}^{(t+k)} \mid \mathcal{F}_t] \iff Y_t \geq V_t^{(t+k)} \because V_t^{(t+k)} = \max\{Y_t, \mathbb{E}[V_{t+1}^{(t+k)} \mid \mathcal{F}_t]\}$.

Simple Stopping Rules

One-Stage Look-Ahead Stopping Time: $\tau_{1\text{-sla}} := \inf\{t \geq 0 \mid Y_t \geq \mathbb{E}[Y_{t+1} \mid \mathcal{F}_t]\}$.

Myopic stopping: stop as soon as it doesn't pay off continuing for one more period.

k-Stage Look-Ahead Stopping Time: $\tau_{k\text{-sla}} := \inf\{t \geq 0 \mid Y_t \geq \mathbb{E}[V_{t+1}^{(t+k)} \mid \mathcal{F}_t]\}$.

Stop if continuing for at most k more periods isn't worthwhile.

Naive: not considering that next period will consider the following k periods instead of $k-1$.

Note: $Y_t \geq \mathbb{E}[V_{t+1}^{(t+k)} \mid \mathcal{F}_t] \iff Y_t \geq V_t^{(t+k)} \because V_t^{(t+k)} = \max\{Y_t, \mathbb{E}[V_{t+1}^{(t+k)} \mid \mathcal{F}_t]\}$.

One-Time Look-Ahead Stopping Time: $\tau_{1\text{-tla}} := \inf\{t \geq 0 \mid Y_t \geq \sup_{\ell>0} \mathbb{E}[Y_{t+\ell} \mid \mathcal{F}_t]\}$.

Continue iff $\exists \ell > 0$: committing to continue ℓ periods more is better than stopping.

Naively committed: $t+\ell$ may decide to continue again.

$\tau_{1\text{-sla}} \leq \tau_{1\text{-tla}}, \tau_{k\text{-sla}} \leq \tau^*$.

Moreover, $\mathbb{E}[Y_{\tau_{1\text{-sla}}}] \leq \mathbb{E}[Y_{\tau_{k\text{-sla}}}], \mathbb{E}[Y_{\tau_{1\text{-sla}}}] \leq \mathbb{E}[Y_{\tau^*}]$.

Monotone Problems

1-sla is optimal in selling a house, satisficing.

When is myopic stopping optimal?

Monotone Problems

1-sla is optimal in selling a house, satisficing.

When is myopic stopping optimal?

Definition

Let $A_t := \{Y_t \geq [Y_{t+1} \mid \mathcal{F}_t]\}$. The stopping problem is monotone if $A_t \subseteq A_{t+1}$ a.s. for any $t = 0, 1, ..., T-1$, where $T \in \mathbb{N} \cup \{\infty\}$.

Monotone Problems

1-sla is optimal in selling a house, satisficing.

When is myopic stopping optimal?

Definition

Let $A_t := \{Y_t \geq [Y_{t+1} \mid \mathcal{F}_t]\}$. The stopping problem is monotone if $A_t \subseteq A_{t+1}$ a.s. for any $t = 0, 1, ..., T - 1$, where $T \in \mathbb{N} \cup \{\infty\}$.

Theorem

In finite horizon monotone stopping problems, $\tau_{\text{1-sla}}$ is optimal.

Monotone Problems

Theorem

In finite horizon monotone stopping problems, $\tau_{1\text{-sla}}$ is optimal.

Proof

Let horizon be T. Earliest optimal stopping $\tau^* := \inf\{t \geq 0 \mid Y_t \geq \mathbb{E}[V_{t+1}^{(T)} \mid \mathcal{F}_t]\}$, with $V_{T+1}^{(T)} = -\infty$ and $V_T^{(T)} = Y_T$.

Monotone Problems

Theorem

In finite horizon monotone stopping problems, $\tau_{\text{1-sla}}$ is optimal.

Proof

Let horizon be T. Earliest optimal stopping $\tau^* := \inf\{t \geq 0 \mid Y_t \geq \mathbb{E}[V_{t+1}^{(T)} \mid \mathcal{F}_t]\}$, with $V_{T+1}^{(T)} = -\infty$ and $V_T^{(T)} = Y_T$.

Bwd induction: $V_t^{(T)} = \max\{Y_t, \mathbb{E}[V_{t+1}^{(T)} \mid \mathcal{F}_t]\}$.

Monotone Problems

Theorem

In finite horizon monotone stopping problems, $\tau_{1\text{-sla}}$ is optimal.

Proof

Let horizon be T. Earliest optimal stopping $\tau^* := \inf\{t \geq 0 \mid Y_t \geq \mathbb{E}[V_{t+1}^{(T)} \mid \mathcal{F}_t]\}$, with $V_{T+1}^{(T)} = -\infty$ and $V_T^{(T)} = Y_T$.

Bwd induction: $V_t^{(T)} = \max\{Y_t, \mathbb{E}[V_{t+1}^{(T)} \mid \mathcal{F}_t]\}$.

Fix $t < T$. Note $\tau_{1\text{-sla}} > t \implies \tau^* > t$. Suppose $\tau_{1\text{-sla}} = t$.

Since $\{\tau_{1\text{-sla}} = t\} = \{Y_t \geq \mathbb{E}[Y_{t+1} \mid \mathcal{F}_t]\} = A_t$ and problem is monotone,

Monotone Problems

Theorem

In finite horizon monotone stopping problems, $\tau_{\text{1-sla}}$ is optimal.

Proof

Let horizon be T. Earliest optimal stopping $\tau^* := \inf\{t \geq 0 \mid Y_t \geq \mathbb{E}[V_{t+1}^{(T)} \mid \mathcal{F}_t]\}$, with $V_{T+1}^{(T)} = -\infty$ and $V_T^{(T)} = Y_T$.

Bwd induction: $V_t^{(T)} = \max\{Y_t, \mathbb{E}[V_{t+1}^{(T)} \mid \mathcal{F}_t]\}$.

Fix $t < T$. Note $\tau_{\text{1-sla}} > t \implies \tau^* > t$. Suppose $\tau_{\text{1-sla}} = t$.

Since $\{\tau_{\text{1-sla}} = t\} = \{Y_t \geq \mathbb{E}[Y_{t+1} \mid \mathcal{F}_t]\} = A_t$ and problem is monotone,

$Y_{T-1} \geq \mathbb{E}[Y_T \mid \mathcal{F}_{T-1}] \implies Y_{T-1} = V_{T-1}^{(T)}$;

Monotone Problems

Theorem

In finite horizon monotone stopping problems, $\tau_{\text{1-sla}}$ is optimal.

Proof

Let horizon be T. Earliest optimal stopping $\tau^* := \inf\{t \geq 0 \mid Y_t \geq \mathbb{E}[V_{t+1}^{(T)} \mid \mathcal{F}_t]\}$, with $V_{T+1}^{(T)} = -\infty$ and $V_T^{(T)} = Y_T$.

Bwd induction: $V_t^{(T)} = \max\{Y_t, \mathbb{E}[V_{t+1}^{(T)} \mid \mathcal{F}_t]\}$.

Fix $t < T$. Note $\tau_{\text{1-sla}} > t \implies \tau^* > t$. Suppose $\tau_{\text{1-sla}} = t$.

Since $\{\tau_{\text{1-sla}} = t\} = \{Y_t \geq \mathbb{E}[Y_{t+1} \mid \mathcal{F}_t]\} = A_t$ and problem is monotone,

$Y_{T-1} \geq \mathbb{E}[Y_T \mid \mathcal{F}_{T-1}] \implies Y_{T-1} = V_{T-1}^{(T)}$;

$Y_{T-2} \geq \mathbb{E}[Y_{T-1} \mid \mathcal{F}_{T-2}] = \mathbb{E}[V_{T-1}^{(T-1)} \mid \mathcal{F}_{T-2}] \implies Y_{T-2} = V_{T-2}^{(T)}$;

Monotone Problems

Theorem

In finite horizon monotone stopping problems, $\tau_{1\text{-sla}}$ is optimal.

Proof

Let horizon be T. Earliest optimal stopping $\tau^* := \inf\{t \geq 0 \mid Y_t \geq \mathbb{E}[V_{t+1}^{(T)} \mid \mathcal{F}_t]\}$, with $V_{T+1}^{(T)} = -\infty$ and $V_T^{(T)} = Y_T$.

Bwd induction: $V_t^{(T)} = \max\{Y_t, \mathbb{E}[V_{t+1}^{(T)} \mid \mathcal{F}_t]\}$.

Fix $t < T$. Note $\tau_{1\text{-sla}} > t \implies \tau^* > t$. Suppose $\tau_{1\text{-sla}} = t$.

Since $\{\tau_{1\text{-sla}} = t\} = \{Y_t \geq \mathbb{E}[Y_{t+1} \mid \mathcal{F}_t]\} = A_t$ and problem is monotone,

$$Y_{T-1} \geq \mathbb{E}[Y_T \mid \mathcal{F}_{T-1}] \implies Y_{T-1} = V_{T-1}^{(T)};$$

$$Y_{T-2} \geq \mathbb{E}[Y_{T-1} \mid \mathcal{F}_{T-2}] = \mathbb{E}[V_{T-1}^{(T-1)} \mid \mathcal{F}_{T-2}] \implies Y_{T-2} = V_{T-2}^{(T)};$$

$$\dots$$

$$Y_t \geq \mathbb{E}[Y_{t+1} \mid \mathcal{F}_t] = \mathbb{E}[V_{t+1}^{(T)} \mid \mathcal{F}_t] \implies Y_t = V_t^{(T)}.$$

Hence, $\tau^* = t$.

Approximating Infinite Horizon by Finite Horizon

Goal: WT use finite horizon result to understand when myopic stopping is optimal in infinite horizon problem.

Approximating Infinite Horizon by Finite Horizon

Goal: WT use finite horizon result to understand when myopic stopping is optimal in infinite horizon problem.

Standing assumptions

(A1) $\mathbb{E}\left[\sup_{t\geq \mathbf{0}} Y_t\right] < \infty$.

(A2) $\lim_{t\to\infty} \mathbb{E}[Y_t] \leq Y_\infty$ a.s.

(A3) $\lim_{t\to\infty} Y_t = Y_\infty$ a.s.

Approximating Infinite Horizon by Finite Horizon

Goal: WT use finite horizon result to understand when myopic stopping is optimal in infinite horizon problem.

Standing assumptions

(A1) $\mathbb{E}\left[\sup_{t\geq 0} Y_t\right] < \infty$.

(A2) $\lim_{t\to\infty} \mathbb{E}[Y_t] \leq Y_\infty$ a.s.

(A3) $\lim_{t\to\infty} Y_t = Y_\infty$ a.s.

Definition

$\{X_t\}$ are uniformly integrable if $\sup_t \mathbb{E}[|X_t|\mathbf{1}_{\{|X_t|>a\}}] \to 0$ as $a \to \infty$.

Conditions for uniform integrability:

1. $\lim_{t\to\infty} \mathbb{E}[|X_t|] = 0$, then $\{X_t\}_t$ is uniform integrable.
2. $\lim_{t\to\infty} \mathbb{E}[|X_t|] = \infty$, then $\{X_t\}_t$ is not uniform integrable.

Approximating Infinite Horizon by Finite Horizon

Standing assumptions: (A1) $\mathbb{E}\left[\sup_{t\geq 0} Y_t\right] < \infty$. (A3) $\lim_{t\to\infty} Y_t = Y_\infty$ a.s.

Theorem

Assume (A1) and (A3). If $Z_t := \sup_{j\geq t} Y_j - Y_t$ is uniformly integrable, then $V_{\mathbf{0}}^{(T)} \to V^*$ as $T \to \infty$.

Approximating Infinite Horizon by Finite Horizon

Standing assumptions: (A1) $\mathbb{E}\left[\sup_{t\geq 0} Y_t\right] < \infty$. (A3) $\lim_{t\to\infty} Y_t = Y_\infty$ a.s.

Theorem

Assume (A1) and (A3). If $Z_t := \sup_{j\geq t} Y_j - Y_t$ is uniformly integrable, then $V_0^{(T)} \to V^*$ as $T \to \infty$.

Proof

Let $A_T := \{T < \tau^* < \infty\}$.

$$
\begin{aligned}
0 \leq V^* - V_0^{(T)} &\leq \mathbb{E}[Y_{\tau^*} - Y_{\tau^* \wedge T}] = \mathbb{E}[\mathbf{1}_{\{A_T\}}(Y_{\tau^*} - Y_T)] + \mathbb{E}[\mathbf{1}_{\{\tau^*=\infty\}}(Y_\infty - Y_T)] \\
&\leq \mathbb{E}[\mathbf{1}_{\{A_T\}} Z_T] + \mathbb{E}[(Y_\infty - Y_T)^+].
\end{aligned}
$$

Approximating Infinite Horizon by Finite Horizon

Standing assumptions: (A1) $\mathbb{E}\left[\sup_{t\geq 0} Y_t\right] < \infty$. (A3) $\lim_{t\to\infty} Y_t = Y_\infty$ a.s.

Theorem

Assume (A1) and (A3). If $Z_t := \sup_{j\geq t} Y_j - Y_t$ is uniformly integrable, then $V_0^{(T)} \to V^*$ as $T \to \infty$.

Proof

Let $A_T := \{T < \tau^* < \infty\}$.

$$0 \leq V^* - V_0^{(T)} \leq \mathbb{E}[Y_{\tau^*} - Y_{\tau^* \wedge T}] = \mathbb{E}[\mathbf{1}_{\{A_T\}}(Y_{\tau^*} - Y_T)] + \mathbb{E}[\mathbf{1}_{\{\tau^*=\infty\}}(Y_\infty - Y_T)]$$
$$\leq \mathbb{E}[\mathbf{1}_{\{A_T\}} Z_T] + \mathbb{E}[(Y_\infty - Y_T)^+].$$

Approximating Infinite Horizon by Finite Horizon

Standing assumptions: (A1) $\mathbb{E}\left[\sup_{t\geq 0} Y_t\right] < \infty$. (A3) $\lim_{t\to\infty} Y_t = Y_\infty$ a.s.

Theorem

Assume (A1) and (A3). If $Z_t := \sup_{j\geq t} Y_j - Y_t$ is uniformly integrable, then $V_0^{(T)} \to V^*$ as $T \to \infty$.

Proof

Let $A_T := \{T < \tau^* < \infty\}$. $0 \leq V^* - V_0^{(T)} \leq \mathbb{E}[\mathbf{1}_{\{A_T\}} Z_T] + \mathbb{E}[(Y_\infty - Y_T)^+]$.

Approximating Infinite Horizon by Finite Horizon

Standing assumptions: (A1) $\mathbb{E}\left[\sup_{t\geq 0} Y_t\right] < \infty$. (A3) $\lim_{t\to\infty} Y_t = Y_\infty$ a.s.

Theorem

Assume (A1) and (A3). If $Z_t := \sup_{j\geq t} Y_j - Y_t$ is uniformly integrable, then $V_0^{(T)} \to V^*$ as $T \to \infty$.

Proof

Let $A_T := \{T < \tau^* < \infty\}$. $0 \leq V^* - V_0^{(T)} \leq \mathbb{E}[\mathbf{1}_{\{A_T\}} Z_T] + \mathbb{E}[(Y_\infty - Y_T)^+]$.

1st term goes to zero.

Approximating Infinite Horizon by Finite Horizon

Standing assumptions: (A1) $\mathbb{E}\left[\sup_{t\geq 0} Y_t\right] < \infty$. (A3) $\lim_{t\to\infty} Y_t = Y_\infty$ a.s.

Theorem

Assume (A1) and (A3). If $Z_t := \sup_{j\geq t} Y_j - Y_t$ is uniformly integrable, then $V_0^{(T)} \to V^*$ as $T \to \infty$.

Proof

Let $A_T := \{T < \tau^* < \infty\}$. $0 \leq V^* - V_0^{(T)} \leq \mathbb{E}[\mathbf{1}_{\{A_T\}} Z_T] + \mathbb{E}[(Y_\infty - Y_T)^+]$.

1st term goes to zero.

Let $q_T = \mathbb{P}(A_T) \to 0$. By uniform integrability of Z_T,

$$\mathbb{E}[\mathbf{1}_{\{A_T\}} Z_T] = \mathbb{E}[\mathbf{1}_{\{A_T\}} \mathbf{1}_{\{Z_T \leq q_T^{-1/2}\}} Z_T] + \mathbb{E}[\mathbf{1}_{\{A_T\}} \mathbf{1}_{\{Z_T > q_T^{-1/2}\}} Z_T] \leq q_T^{1/2} + \mathbb{E}[\mathbf{1}_{\{Z_T > q_T^{-1/2}\}} Z_T] \to 0.$$

Approximating Infinite Horizon by Finite Horizon

Standing assumptions: (A1) $\mathbb{E}\left[\sup_{t\geq 0} Y_t\right] < \infty$. (A3) $\lim_{t\to\infty} Y_t = Y_\infty$ a.s.

Theorem

Assume (A1) and (A3). If $Z_t := \sup_{j\geq t} Y_j - Y_t$ is uniformly integrable, then $V_0^{(T)} \to V^*$ as $T \to \infty$.

Proof

Let $A_T := \{T < \tau^* < \infty\}$. $0 \leq V^* - V_0^{(T)} \leq \mathbb{E}[\mathbf{1}_{\{A_T\}} Z_T] + \mathbb{E}[(Y_\infty - Y_T)^+]$.

1st term goes to zero.

Let $q_T = \mathbb{P}(A_T) \to 0$. By uniform integrability of Z_T,

$$\mathbb{E}[\mathbf{1}_{\{A_T\}} Z_T] = \mathbb{E}[\mathbf{1}_{\{A_T\}} \mathbf{1}_{\{Z_T \leq q_T^{-1/2}\}} Z_T] + \mathbb{E}[\mathbf{1}_{\{A_T\}} \mathbf{1}_{\{Z_T > q_T^{-1/2}\}} Z_T] \leq q_T^{1/2} + \mathbb{E}[\mathbf{1}_{\{Z_T > q_T^{-1/2}\}} Z_T] \to 0.$$

Let $p_T = \mathbb{P}(Y_\infty - Y_T > q_T)^{-1/2} \to \infty$.

By uniform integrability of Z_T, $\mathbb{E}[\mathbf{1}_{\{A_T\}} Z_T] = \mathbb{E}[\mathbf{1}_{\{A_T\}} \mathbf{1}_{\{Z_T \leq q_T\}} Z_T] + \mathbb{E}[\mathbf{1}_{\{A_T\}} \mathbf{1}_{\{Z_T > q_T\}} Z_T] \leq q_T \mathbb{P}(A_T) + \mathbb{E}[\mathbf{1}_{\{Z_T > q_T\}} Z_T] \to 0$.

Approximating Infinite Horizon by Finite Horizon

Standing assumptions: (A1) $\mathbb{E}\left[\sup_{t\geq 0} Y_t\right] < \infty$.   (A3) $\lim_{t\to\infty} Y_t = Y_\infty$ a.s.

Theorem

Assume (A1) and (A3). If $Z_t := \sup_{j\geq t} Y_j - Y_t$ is uniformly integrable, then $V_{\mathbf{0}}^{(T)} \to V^*$ as $T \to \infty$.

Proof

Let $A_T := \{T < \tau^* < \infty\}$. $0 \leq V^* - V_{\mathbf{0}}^{(T)} \leq \mathbb{E}[\mathbf{1}_{\{A_T\}} Z_T] + \mathbb{E}[(Y_\infty - Y_T)^+]$.

1st term goes to zero.

Approximating Infinite Horizon by Finite Horizon

Standing assumptions: (A1) $\mathbb{E}\left[\sup_{t\geq 0} Y_t\right] < \infty$. (A3) $\lim_{t\to\infty} Y_t = Y_\infty$ a.s.

Theorem

Assume (A1) and (A3). If $Z_t := \sup_{j\geq t} Y_j - Y_t$ is uniformly integrable, then $V_0^{(T)} \to V^*$ as $T \to \infty$.

Proof

Let $A_T := \{T < \tau^* < \infty\}$. $0 \leq V^* - V_0^{(T)} \leq \mathbb{E}[\mathbf{1}_{\{A_T\}} Z_T] + \mathbb{E}[(Y_\infty - Y_T)^+]$.

1st term goes to zero.

2nd term goes to zero. Let $\varepsilon_T \to 0 : \mathbb{P}(Y_\infty - Y_T > \varepsilon_T) \to 0$.

Approximating Infinite Horizon by Finite Horizon

Standing assumptions: (A1) $\mathbb{E}\left[\sup_{t\geq 0} Y_t\right] < \infty$. (A3) $\lim_{t\to\infty} Y_t = Y_\infty$ a.s.

Theorem

Assume (A1) and (A3). If $Z_t := \sup_{j\geq t} Y_j - Y_t$ is uniformly integrable, then $V_0^{(T)} \to V^*$ as $T \to \infty$.

Proof

Let $A_T := \{T < \tau^* < \infty\}$. $0 \leq V^* - V_0^{(T)} \leq \mathbb{E}[\mathbf{1}_{\{A_T\}} Z_T] + \mathbb{E}[(Y_\infty - Y_T)^+]$.

1st term goes to zero.

2nd term goes to zero. Let $\varepsilon_T \to 0 : \mathbb{P}(Y_\infty - Y_T > \varepsilon_T) \to 0$.

$\mathbb{E}[(Y_\infty - Y_T)^+] = \mathbb{E}[\mathbf{1}_{\{Y_\infty - Y_t \leq \varepsilon_T\}}(Y_\infty - Y_T)^+] + \mathbb{E}[\mathbf{1}_{\{Y_\infty - Y_T > \varepsilon_T\}}(Y_\infty - Y_T)^+] \leq \varepsilon_T + \mathbb{E}[\mathbf{1}_{\{Y_\infty - Y_T > \varepsilon_T\}} Z_T]$.

$\mathbb{E}[\mathbf{1}_{\{Y_\infty - Y_T > \varepsilon_T\}} Z_T] \to 0$ follows by similar argument as before for 1st term.

Approximating Infinite Horizon by Finite Horizon

Standing assumptions: (A1) $\mathbb{E}\left[\sup_{t\geq 0} Y_t\right] < \infty$. (A3) $\lim_{t\to\infty} Y_t = Y_\infty$ a.s.

Theorem

Assume (A1) and (A3). If $Z_t := \sup_{j\geq t} Y_j - Y_t$ is uniformly integrable, then $V_0^{(T)} \to V^*$ as $T \to \infty$.

Corollary

Assume (A3). If $Y_t := B_t - C_t$, where $\mathbb{E}[\sup_t |B_t|] < \infty$ and $C_t \geq 0$ and nondecreasing a.s., then (A1) holds and $V_0^{(T)} \to V^*$.

Approximating Infinite Horizon by Finite Horizon

Standing assumptions: (A1) $\mathbb{E}\left[\sup_{t\geq 0} Y_t\right] < \infty$. (A3) $\lim_{t\to\infty} Y_t = Y_\infty$ a.s.

Theorem

Assume (A1) and (A3). If $Z_t := \sup_{j\geq t} Y_j - Y_t$ is uniformly integrable, then $V_0^{(T)} \to V^*$ as $T \to \infty$.

Corollary

Assume (A3). If $Y_t := B_t - C_t$, where $\mathbb{E}[\sup_t |B_t|] < \infty$ and $C_t \geq 0$ and nondecreasing a.s., then (A1) holds and $V_0^{(T)} \to V^*$.

Proof

$\mathbb{E}\left[\sup_{t\geq 0} Y_t\right] \leq \mathbb{E}\left[\sup_{t\geq 0} |B_t|\right] < \infty \implies$ (A1) holds.

For $j \geq t$, $Y_j - Y_t = B_j - B_t + (C_t - C_j) \leq B_j - B_t$.

$0 \leq Z_t := \sup_{j\geq t} Y_j - Y_t \leq 2 \sup_t |B_t| =: B'$.

$\mathbb{E}[B'] < \infty$, hence $\mathbb{E}[1_{\{|Z_t|>a\}} |Z_t|] \leq \mathbb{E}[1_{\{B'>a\}} B'] \to 0$ and Z_t is uniformly integrable.

Overview

Stopping and Choosing: Selling a House

Accept best offer M_t or continue waiting with a per period cost of c.

Interpretation:

- Selling a house/asset: offers $X_t \geq 0$ come in, council tax/management fees c;
  $Y_t := M_t - ct$, where $M_t := \max_{s \leq t} X_t$.
- Same as satisficing, just take $T = \infty$.

$X_t \sim F$, iid; F continuous, strictly increasing, with finite 2nd moment.

Stopping and Choosing: Selling a House

Accept and get $M_t - tc$;

Refuse and pay c and wait for one more offer tomorrow.

Markov problem; state variable = Y_t

Stopping and Choosing: Selling a House

- Accept and get $M_t - tc$;
  - Refuse and pay c and wait for one more offer tomorrow.
- Markov problem; state variable = Y_t
- Set up Bellman equation; $V(Y_t) = \max\{Y_t, \mathbb{E}[V(Y_{t+1})] - c\}$.
- Define $V_t := V(Y_t)$; $\mathbb{E}[V(Y_t)]$ now depends on t!
  - Simple derivation from before no longer works...

Stopping and Choosing: Selling a House

Accept and get $M_t - tc$;

Refuse and pay c and wait for one more offer tomorrow.

Markov problem; state variable = Y_t

Set up Bellman equation; $V(Y_t) = \max\{Y_t, \mathbb{E}[V(Y_{t+1})] - c\}$.

Define $V_t := V(Y_t)$; $\mathbb{E}[V(Y_t)]$ now depends on t!

Simple derivation from before no longer works...

But this is a **monotone problem**:

$Y_t \geq \mathbb{E}[Y_{t+1} \mid \mathcal{F}_t] \iff Y_t \geq \mathbb{E}[\max\{Y_t, X_{t+1} - tc\} \mid \mathcal{F}_t] - c \iff c \geq \mathbb{E}[(X_0 - (Y_t + tc))^+ \mid \mathcal{F}_t]$.

Since $Y_t + tc$ is increasing in t, $\{Y_t \geq \mathbb{E}[Y_{t+1} \mid \mathcal{F}_t]\} \subseteq \{Y_{t+\ell} \geq \mathbb{E}[Y_{t+\ell+1} \mid \mathcal{F}_{t+\ell}]\}$ for any $t \geq 0$ and $\ell \geq 0$.

Check conditions for approximation: (A1), (A3), and UI...

Stopping and Choosing: Selling a House

Theorem

Let $X, X_1, X_2, \ldots,$ be iid, $c > 0$, and $Y_t = X_t - tc$ or $Y_t = \max_{s \leq t} X_s - tc$.

- If $\mathbb{E}[X^+] < \infty$, then $\sup_t Y_t < \infty$ a.s. and $Y_t \to -\infty$ a.s.
- If $\mathbb{E}[(X^+)^2] < \infty$, then $\mathbb{E}[\sup_t Y_t] < \infty$.

Proof

See the proof to Theorem 1 in Ferguson (2008, Ch. 4, Appendix).

Stopping and Choosing: Selling a House

Theorem

Let $X, X_1, X_2, \ldots,$ be iid, $c > 0$, and $Y_t = X_t - tc$ or $Y_t = \max_{s \leq t} X_s - tc$.

If $\mathbb{E}[X^+] < \infty$, then $\sup_t Y_t < \infty$ a.s. and $Y_t \to -\infty$ a.s.

If $\mathbb{E}[(X^+)^2] < \infty$, then $\mathbb{E}[\sup_t Y_t] < \infty$.

Proof

See the proof to Theorem 1 in Ferguson (2008, Ch. 4, Appendix).

(A1): $\mathbb{E}[X^+] < \infty \implies \mathbb{E}\left[\sup_{t \geq 0} Y_t\right] < \infty$. Check.

Stopping and Choosing: Selling a House

Theorem

Let $X, X_1, X_2, \ldots,$ be iid, $c > 0$, and $Y_t = X_t - tc$ or $Y_t = \max_{s \le t} X_s - tc$.

If $\mathbb{E}[X^+] < \infty$, then $\sup_t Y_t < \infty$ a.s. and $Y_t \to -\infty$ a.s.

If $\mathbb{E}[(X^+)^2] < \infty$, then $\mathbb{E}[\sup_t Y_t] < \infty$.

Proof

See the proof to Theorem 1 in Ferguson (2008, Ch. 4, Appendix).

(A1): $\mathbb{E}[X^+] < \infty \implies \mathbb{E}\left[\sup_{t \ge 0} Y_t\right] < \infty$. Check.

(A3): Define $Y_\infty := -\infty$. $\mathbb{E}[X^+] < \infty \implies Y_t \to Y_\infty$. Check.

Stopping and Choosing: Selling a House

Theorem

Let $X, X_1, X_2, \ldots,$ be iid, $c > 0$, and $Y_t = X_t - tc$ or $Y_t = \max_{s \leq t} X_s - tc$.

If $\mathbb{E}[X^+] < \infty$, then $\sup_t Y_t < \infty$ a.s. and $Y_t \to -\infty$ a.s.

If $\mathbb{E}[(X^+)^2] < \infty$, then $\mathbb{E}[\sup_t Y_t] < \infty$.

Proof

See the proof to Theorem 1 in Ferguson (2008, Ch. 4, Appendix).

(A1): $\mathbb{E}[X^+] < \infty \implies \mathbb{E}\left[\sup_{t \geq 0} Y_t\right] < \infty$. Check.

(A3): Define $Y_\infty := -\infty$. $\mathbb{E}[X^+] < \infty \implies Y_t \to Y_\infty$. Check.

Uniform integrability: $Z_t := \sup_{j \geq t} Y_j - Y_t = \sup_{j \geq t} (M_j - M_t)^+ - jc$.

Stopping and Choosing: Selling a House

Theorem

Let $X, X_1, X_2, \ldots,$ be iid, $c > 0$, and $Y_t = X_t - tc$ or $Y_t = \max_{s \le t} X_s - tc$.

If $\mathbb{E}[X^+] < \infty$, then $\sup_t Y_t < \infty$ a.s. and $Y_t \to -\infty$ a.s.

If $\mathbb{E}[(X^+)^2] < \infty$, then $\mathbb{E}[\sup_t Y_t] < \infty$.

Proof

See the proof to Theorem 1 in Ferguson (2008, Ch. 4, Appendix).

(A1): $\mathbb{E}[X^+] < \infty \implies \mathbb{E}\left[\sup_{t \ge 0} Y_t\right] < \infty$. Check.

(A3): Define $Y_\infty := -\infty$. $\mathbb{E}[X^+] < \infty \implies Y_t \to Y_\infty$. Check.

Uniform integrability: $Z_t := \sup_{j \ge t} Y_j - Y_t = \sup_{j \ge t}(M_j - M_t)^+ - jc$.

Note $\mathbb{E}[Z_t] = \mathbb{E}\left[\mathbb{E}\left[\sup_{j \ge 0} M'_j - jc \mid M_t\right]\right]$ where $M'_j := \max_{s \le j} X'_s$ and $X' := (X - M_t)^+$.

Stopping and Choosing: Selling a House

Theorem

Let $X, X_1, X_2, \ldots,$ be iid, $c > 0$, and $Y_t = X_t - tc$ or $Y_t = \max_{s \le t} X_s - tc$.

If $\mathbb{E}[X^+] < \infty$, then $\sup_t Y_t < \infty$ a.s. and $Y_t \to -\infty$ a.s.

If $\mathbb{E}[(X^+)^2] < \infty$, then $\mathbb{E}[\sup_t Y_t] < \infty$.

Proof

See the proof to Theorem 1 in Ferguson (2008, Ch. 4, Appendix).

(A1): $\mathbb{E}[X^+] < \infty \implies \mathbb{E}\left[\sup_{t \ge 0} Y_t\right] < \infty$. Check.

(A3): Define $Y_\infty := -\infty$. $\mathbb{E}[X^+] < \infty \implies Y_t \to Y_\infty$. Check.

Uniform integrability: $Z_t := \sup_{j \ge t} Y_j - Y_t = \sup_{j \ge t}(M_j - M_t)^+ - jc$.

Note $\mathbb{E}[Z_t] = \mathbb{E}\left[\mathbb{E}\left[\sup_{j \ge 0} M_j' - jc \mid M_t\right]\right]$ where $M_j' := \max_{s \le j} X_s'$ and $X' := (X - M_t)^+$.

$\mathbb{E}[({X'}^+)^2 \mid M_t] = \mathbb{E}[{X'}^2 \mid M_t] < \infty \implies \mathbb{E}[\sup_{j \ge 0} M_j' - jc \mid M_t] < \infty \implies \mathbb{E}[Z_t] < \infty$.

Stopping and Choosing: Selling a House

Theorem

Let $X, X_1, X_2, ...,$ be iid, $c > 0$, and $Y_t = X_t - tc$ or $Y_t = \max_{s \leq t} X_s - tc$.

If $\mathbb{E}[X^+] < \infty$, then $\sup_t Y_t < \infty$ a.s. and $Y_t \to -\infty$ a.s.

If $\mathbb{E}[(X^+)^2] < \infty$, then $\mathbb{E}[\sup_t Y_t] < \infty$.

Proof

See the proof to Theorem 1 in Ferguson (2008, Ch. 4, Appendix).

(A1): $\mathbb{E}[X^+] < \infty \implies \mathbb{E}\left[\sup_{t \geq 0} Y_t\right] < \infty$. Check.

(A3): Define $Y_\infty := -\infty$. $\mathbb{E}[X^+] < \infty \implies Y_t \to Y_\infty$. Check.

Uniform integrability: $Z_t := \sup_{j \geq t} Y_j - Y_t = \sup_{j \geq t}(M_j - M_t)^+ - jc$.

Note $\mathbb{E}[Z_t] = \mathbb{E}\left[\mathbb{E}\left[\sup_{j \geq 0} M_j' - jc \mid M_t\right]\right]$ where $M_j' := \max_{s \leq j} X_s'$ and $X' := (X - M_t)^+$.

$\mathbb{E}[({X'}^+)^2 \mid M_t] = \mathbb{E}[{X'}^2 \mid M_t] < \infty \implies \mathbb{E}[\sup_{j \geq 0} M_j' - jc \mid M_t] < \infty \implies \mathbb{E}[Z_t] < \infty$.

$(X - M_t)^+ \xrightarrow{d} \delta_0$ as $t \to \infty \implies \mathbb{E}[Z_t] \to 0$.

Stopping and Choosing: Selling a House

Theorem

Let $X, X_1, X_2, \ldots,$ be iid, $c > 0$, and $Y_t = X_t - tc$ or $Y_t = \max_{s \leq t} X_s - tc$.

If $\mathbb{E}[X^+] < \infty$, then $\sup_t Y_t < \infty$ a.s. and $Y_t \to -\infty$ a.s.

If $\mathbb{E}[(X^+)^2] < \infty$, then $\mathbb{E}[\sup_t Y_t] < \infty$.

Proof

See the proof to Theorem 1 in Ferguson (2008, Ch. 4, Appendix).

(A1): $\mathbb{E}[X^+] < \infty \implies \mathbb{E}\left[\sup_{t \geq 0} Y_t\right] < \infty$. Check.

(A3): Define $Y_\infty := -\infty$. $\mathbb{E}[X^+] < \infty \implies Y_t \to Y_\infty$. Check.

Uniform integrability: $Z_t := \sup_{j \geq t} Y_j - Y_t = \sup_{j \geq t}(M_j - M_t)^+ - jc$.

Note $\mathbb{E}[Z_t] = \mathbb{E}\left[\mathbb{E}\left[\sup_{j \geq 0} M'_j - jc \mid M_t\right]\right]$ where $M'_j := \max_{s \leq j} X'_s$ and $X' := (X - M_t)^+$.

$\mathbb{E}[(X'^+)^2 \mid M_t] = \mathbb{E}[X'^2 \mid M_t] < \infty \implies \mathbb{E}[\sup_{j \geq 0} M'_j - jc \mid M_t] < \infty \implies \mathbb{E}[Z_t] < \infty$.

$(X - M_t)^+ \xrightarrow{d} \delta_0$ as $t \to \infty \implies \mathbb{E}[Z_t] \to 0$.

$\implies \sup_t \mathbb{E}[Z_t] < \infty \implies \sup_t \mathbb{E}[Z_t \mathbf{1}_{\{Z_t > a\}}] \to 0$ as $a \to \infty$. Check.

Stopping and Choosing: Selling a House

Theorem

Let $X, X_1, X_2, \ldots,$ be iid, $c > 0$, and $Y_t = X_t - tc$ or $Y_t = \max_{s \leq t} X_s - tc$.

If $\mathbb{E}[X^+] < \infty$, then $\sup_t Y_t < \infty$ a.s. and $Y_t \to -\infty$ a.s.

If $\mathbb{E}[(X^+)^2] < \infty$, then $\mathbb{E}[\sup_t Y_t] < \infty$.

Proof

See the proof to Theorem 1 in Ferguson (2008, Ch. 4, Appendix).

(A1): $\mathbb{E}[X^+] < \infty \implies \mathbb{E}\left[\sup_{t \geq 0} Y_t\right] < \infty$. Check.

(A3): Define $Y_\infty := -\infty$. $\mathbb{E}[X^+] < \infty \implies Y_t \to Y_\infty$. Check.

Uniform integrability: $Z_t := \sup_{j \geq t} Y_j - Y_t = \sup_{j \geq t}(M_j - M_t)^+ - jc$.

Note $\mathbb{E}[Z_t] = \mathbb{E}\left[\mathbb{E}\left[\sup_{j \geq 0} M'_j - jc \mid M_t\right]\right]$ where $M'_j := \max_{s \leq j} X'_s$ and $X' := (X - M_t)^+$.

$\mathbb{E}[(X'^+)^2 \mid M_t] = \mathbb{E}[X'^2 \mid M_t] < \infty \implies \mathbb{E}[\sup_{j \geq 0} M'_j - jc \mid M_t] < \infty \implies \mathbb{E}[Z_t] < \infty$.

$(X - M_t)^+ \xrightarrow{d} \delta_0$ as $t \to \infty \implies \mathbb{E}[Z_t] \to 0$.

$\implies \sup_t \mathbb{E}[Z_t] < \infty \implies \sup_t \mathbb{E}[Z_t \mathbf{1}_{\{Z_t > a\}}] \to 0$ as $a \to \infty$. Check.

Conclude 1-sla is still optimal with infinite horizon!

Stopping and Choosing: Selling a House

Selling a house with TIOLI offers:

$Y_t := X_t - tc, X_t \sim F$ iid.

This is *not* a monotone problem!

Stopping and Choosing: Selling a House

Selling a house with TIOLI offers:

$Y_t := X_t - tc, X_t \sim F$ iid.

This is *not* a monotone problem!

Selling a house with distributional uncertainty:

$Y_t := M_t - tc, X_t \sim F(\cdot \mid \theta)$ iid, but θ unknown, $\theta \sim P$.

Let $\mathbb{E}[\mathbf{1}_{\{X_t \leq \cdot\}} \mid \mathcal{F}_t] = F_t$ and suppose that $F_t = \frac{\alpha_0}{\alpha_0 + t} F_0 + \frac{t}{\alpha_0 + t} \hat{F}_t$, where $\hat{F}_t$ is ECDF, $\alpha_0 > 0$, and F_0 has finite 2nd moment. (E.g., Dirichlet process prior.)

This is a monotone problem and 1-sla is still optimal. Prove it!

Overview

The Diamond (1971 JPE) Model (adapted)

Foundational model of price search.

The Diamond (1971 JPE) Model (adapted)

Foundational model of price search.

Environment

- N identical sellers; homogenous good; zero marginal cost (normalisation).
- Identical mass 1 of consumers; unit demand (generalises).
- Known valuation $v > 0$. Value from purchase at price $\hat{p}$ is $v - \hat{p}$.

The Diamond (1971 JPE) Model (adapted)

Foundational model of price search.

Environment

N identical sellers; homogenous good; zero marginal cost (normalisation).

Identical mass 1 of consumers; unit demand (generalises).

Known valuation $v > 0$. Value from purchase at price $\hat{p}$ is $v - \hat{p}$.

Timing

Sellers set prices $p = \{p^n\} \subset \mathbb{R}_+$.

Consumer knows empirical distribution of prices,
but not which seller sets which price.

Consumer learns price of seller n only by visiting seller.
Visit bears a cost $c > 0$. (visit, browse, ask for a quote, etc.)

Sellers selected to visit uniformly at random (among those not yet visited).

Following each visit, consumer can either choose to buy good from one of the sellers they visited or to learn the price of another seller.

The Diamond (1971 JPE) Model (adapted)

Foundational model of price search.

Environment

N identical sellers; homogenous good; zero marginal cost (normalisation).

Identical mass 1 of consumers; unit demand (generalises).

Known valuation $v > \mathbf{0}$. Value from purchase at price $\hat{p}$ is $v - \hat{p}$.

Timing

Sellers set prices $p = \{p^n\} \subset \mathbb{R}_+$.

Consumer knows empirical distribution of prices,
but not which seller sets which price.

Consumer learns price of seller n only by visiting seller.
Visit bears a cost $c > \mathbf{0}$. (visit, browse, ask for a quote, etc.)

Sellers selected to visit uniformly at random (among those not yet visited).

Following each visit, consumer can either choose to buy good from one of the sellers they visited or to learn the price of another seller.

Key Features

Uncertainty over *prices*, not match values.

Solving for Equilibrium Prices

Notation

$n_t \in \{1, ..., N\}$: seller sampled at t.

$S_t := \{n_1, ..., n_t\}$: sellers sampled by t (consideration set).

$N_t := \{1, ..., N\} \setminus S_t$: sellers not yet sampled by t.

Solving for Equilibrium Prices

Notation

$n_t \in \{1, ..., N\}$: seller sampled at t.

$S_t := \{n_1, ..., n_t\}$: sellers sampled by t (consideration set).

$N_t := \{1, ..., N\} \setminus S_t$: sellers not yet sampled by t.

Solving for Equilibrium Prices

Notation

$n_t \in \{1, ..., N\}$: seller sampled at t.

$S_t := \{n_1, ..., n_t\}$: sellers sampled by t (consideration set).

$N_t := \{1, ..., N\} \setminus S_t$: sellers not yet sampled by t.

$n_t \sim U(N_{t-1})$: sellers sampled uniformly at random.

$p_t := p^{n_t} \sim F_t$: price of sampled seller.

Solving for Equilibrium Prices

Notation

$n_t \in \{1, ..., N\}$: seller sampled at t.

$S_t := \{n_1, ..., n_t\}$: sellers sampled by t (consideration set).

$N_t := \{1, ..., N\} \setminus S_t$: sellers not yet sampled by t.

$n_t \sim U(N_{t-1})$: sellers sampled uniformly at random.

$p_t := p^{n_t} \sim F_t$: price of sampled seller.

$X_t := v - p_t$; $M_t := \max_{s \leq t} v - p_s$; $Y_t := M_t - tc$; $V_t := \operatorname{ess\,sup}_{\tau \geq t} \mathbb{E}[Y_\tau \mid \mathcal{F}_t]$.

Solving for Equilibrium Prices

Notation

$n_t \in \{1, ..., N\}$: seller sampled at t.

$S_t := \{n_1, ..., n_t\}$: sellers sampled by t (consideration set).

$N_t := \{1, ..., N\} \setminus S_t$: sellers not yet sampled by t.

$n_t \sim U(N_{t-1})$: sellers sampled uniformly at random.

$p_t := p^{n_t} \sim F_t$: price of sampled seller.

$X_t := v - p_t;\quad M_t := \max_{s \leq t} v - p_s;\quad Y_t := M_t - tc;\quad V_t := \operatorname{ess\,sup}_{\tau \geq t} \mathbb{E}[Y_\tau \mid \mathcal{F}_t]$.

Fix prices and label sellers: $\underline{p} = p^1 \leq \cdots \leq p^N = \overline{p}$.

τ: optimal stopping by consumer.

Note: $Y_t = v - \underline{p} - tc \implies \tau \leq t$.

Claim

$\underline{p} = \overline{p} \leq v$.

Solving for Equilibrium Prices

Claim

$\underline{p} = \overline{p} \leq v$.

Suppose not. If $\overline{p} > v$, then seller N has strict incentive to lower price to $v - \varepsilon$ for some small enough $\varepsilon > 0$. Then $\underline{p} = p^1 < p^N = \overline{p}$.

Solving for Equilibrium Prices

Claim

$\underline{p} = \overline{p} \leq v$.

Suppose not. If $\overline{p} > v$, then seller N has strict incentive to lower price to $v - \varepsilon$ for some small enough $\varepsilon > 0$. Then $\underline{p} = p^1 < p^N = \overline{p}$.

WTS that seller 1 can increase profits by increasing the price.

Prob. purchase 1 = $\mathbb{P}(n_{t+1} = 1 \text{ and } \tau > t)$.

Solving for Equilibrium Prices

Claim

$\underline{p} = \overline{p} \leq v$.

Suppose not. If $\overline{p} > v$, then seller N has strict incentive to lower price to $v - \varepsilon$ for some small enough $\varepsilon > 0$. Then $\underline{p} = p^1 < p^N = \overline{p}$.

WTS that seller 1 can increase profits by increasing the price.

Prob. purchase 1 = $\mathbb{P}(n_{t+1} = 1 \text{ and } \tau > t)$.

Case 1. At $\{n_{t+1} = 1\} \cap \{\tau \leq t\}$, seller 1 gets zero, so increasing its price does not harm profits.

Solving for Equilibrium Prices

Claim

$\underline{p} = \overline{p} \leq v$.

Suppose not. If $\overline{p} > v$, then seller N has strict incentive to lower price to $v - \varepsilon$ for some small enough $\varepsilon > 0$. Then $\underline{p} = p^1 < p^N = \overline{p}$.

WTS that seller 1 can increase profits by increasing the price.

Prob. purchase 1 = $\mathbb{P}(n_{t+1} = 1 \text{ and } \tau > t)$.

Case 1. At $\{n_{t+1} = 1\} \cap \{\tau \leq t\}$, seller 1 gets zero, so increasing its price does not harm profits.

Case 2. At $\{n_1 = 1\}$, $\tau = 1$. Seller 1 can increase price by $c/2$ while still deterring further search:
continuation value is at best $v - \underline{p} - 2c < v - (\underline{p} + c/2) - c$ = value of stopping and paying $\underline{p} + c/$.

Solving for Equilibrium Prices

Claim

$\underline{p} = \overline{p} \leq v$.

Suppose not. If $\overline{p} > v$, then seller N has strict incentive to lower price to $v - \varepsilon$ for some small enough $\varepsilon > 0$. Then $\underline{p} = p^1 < p^N = \overline{p}$.

WTS that seller 1 can increase profits by increasing the price.

Prob. purchase 1 = $\mathbb{P}(n_{t+1} = 1 \text{ and } \tau > t)$.

Case 1. At $\{n_{t+1} = 1\} \cap \{\tau \leq t\}$, seller 1 gets zero, so increasing its price does not harm profits.

Case 2. At $\{n_1 = 1\}$, $\tau = 1$. Seller 1 can increase price by $c/2$ while still deterring further search:
continuation value is at best $v - \underline{p} - 2c < v - (\underline{p} + c/2) - c$ = value of stopping and paying $\underline{p} + c/$.

Case 3. At $\{n_{t+1} = 1\} \cap \{\tau > t\}$ $t \geq 1$, it must be that $M_t < v - \underline{p}$.
$\tau > t$ implies $\mathbb{E}[V_{t+1} \mid \mathcal{F}_t] - Y_t =: \varepsilon(M_t; p) > 0$.
More: conditional on $\tau > t$, $\exists$ finitely many values possible for $M_t \in \hat{M} := \{v - \hat{p}, \hat{p} \in$

Solving for Equilibrium Prices

Claim

$\underline{p} = \overline{p} \leq v$.

Suppose not. If $\overline{p} > v$, then seller N has strict incentive to lower price to $v - \varepsilon$ for some small enough $\varepsilon > 0$. Then $\underline{p} = p^1 < p^N = \overline{p}$.

WTS that seller 1 can increase profits by increasing the price.

Prob. purchase 1 = $\mathbb{P}(n_{t+1} = 1 \text{ and } \tau > t)$.

Case 1. At $\{n_{t+1} = 1\} \cap \{\tau \leq t\}$, seller 1 gets zero, so increasing its price does not harm profits.

Case 2. At $\{n_1 = 1\}$, $\tau = 1$. Seller 1 can increase price by $c/2$ while still deterring further search:
continuation value is at best $v - \underline{p} - 2c < v - (\underline{p} + c/2) - c$ = value of stopping and paying $\underline{p} + c/$.

Case 3. At $\{n_{t+1} = 1\} \cap \{\tau > t\}$ $t \geq 1$, it must be that $M_t < v - \underline{p}$.
$\tau > t$ implies $\mathbb{E}[V_{t+1} \mid \mathcal{F}_t] - Y_t =: \varepsilon(M_t; p) > 0$.
More: conditional on $\tau > t$, $\exists$ finitely many values possible for $M_t \in \hat{M} := \{v - \hat{p}, \hat{p} \in$

Solving for Equilibrium Prices

Claim

$\underline{p} = \overline{p} \leq v$.

Suppose not. If $\overline{p} > v$, then seller N has strict incentive to lower price to $v - \varepsilon$ for some small enough $\varepsilon > 0$. Then $\underline{p} = p^1 < p^N = \overline{p}$.

WTS that seller 1 can increase profits by increasing the price.

Prob. purchase 1 = $\mathbb{P}(n_{t+1} = 1 \text{ and } \tau > t)$.

Case 1. At $\{n_{t+1} = 1\} \cap \{\tau \leq t\}$, seller 1 gets zero, so increasing its price does not harm profits.

Case 2. At $\{n_1 = 1\}$, $\tau = 1$. Seller 1 can increase price by $c/2$ while still deterring further search:
continuation value is at best $v - \underline{p} - 2c < v - (\underline{p} + c/2) - c$ = value of stopping and paying $\underline{p} + c/$.

Case 3. At $\{n_{t+1} = 1\} \cap \{\tau > t\}$ $t \geq 1$, it must be that $M_t < v - \underline{p}$.
$\tau > t$ implies $\mathbb{E}[V_{t+1} \mid \mathcal{F}_t] - Y_t =: \varepsilon(M_t; p) > 0$.

Solving for Equilibrium Prices

Claim

$\underline{p} = \overline{p} = v$.

Suppose $\underline{p} = \overline{p} < v$. WTS that seller 1 can increase profits by increasing the price.

Purchase 1 only if $n_1 = \mathbf{1}$.

Solving for Equilibrium Prices

Claim

$\underline{p} = \overline{p} = v$.

Suppose $\underline{p} = \overline{p} < v$. WTS that seller 1 can increase profits by increasing the price.

Purchase 1 only if $n_1 = \mathbf{1}$.

Case 1. At $\{n_1 \neq \mathbf{1}\}$, seller 1 gets zero, so increasing its price does not harm profits.

Solving for Equilibrium Prices

Claim

$\underline{p} = \overline{p} = v$.

Suppose $\underline{p} = \overline{p} < v$. WTS that seller 1 can increase profits by increasing the price.

Purchase 1 only if $n_1 = 1$.

Case 1. At $\{n_1 \neq 1\}$, seller 1 gets zero, so increasing its price does not harm profits.

Case 2. At $\{n_1 = 1\}$. Seller 1 can increase price by $c/2$ while still deterring further search: continuation value is $v - \underline{p} - 2c < v - (\underline{p} + c/2) - c$ = value of stopping and paying $\underline{p} + c/$.

Solving for Equilibrium Prices

Claim

$\underline{p} = \overline{p} = v$.

Suppose $\underline{p} = \overline{p} < v$. WTS that seller 1 can increase profits by increasing the price.

Purchase 1 only if $n_1 = 1$.

Case 1. At $\{n_1 \neq 1\}$, seller 1 gets zero, so increasing its price does not harm profits.

Case 2. At $\{n_1 = 1\}$. Seller 1 can increase price by $c/2$ while still deterring further search: continuation value is $v - \underline{p} - 2c < v - (\underline{p} + c/2) - c$ = value of stopping and paying $\underline{p} + c/$.

Increasing price is strictly profitable deviation.

Diamond's Paradox

Implications

"Who cares about search costs in the digital age? Such costs are minute!"

Diamond's Paradox

Implications

"Who cares about search costs in the digital age? Such costs are minute!"

The Paradox

Any arbitrarily small search cost ($c > 0$) causes the market outcome to jump discontinuously from competitive Bertrand outcome ($p = 0$) to full monopoly outcome ($p = V$)! Slightest search friction destroys all price competition.

Overview

Some jargon:

- *With recall:* possibility of choosing any of the samples thus far. *Without recall:* can only choose current element or sample again.
- *Without replacement:* samples are all distinct. *Without replacement:* can resample previously observed sample.
- *Undirected search:* fixed order. *Directed search:* choose the order (more next lecture).

References I

Fundamentals

Stigler (1961 JPE). The Economics of Information. https://doi.org/10.1086/258464.

McCall (1970 QJE). Economics of Information and Job Search. https://doi.org/10.2307/1879403.

Diamon (1971 JET). A model of price adjustment. https://doi.org/10.1016/0022-0531(71)90013-5.

Rothschild (1974 JPE). Searching for the Lowest Price When the Distribution of Prices Is Known. https://doi.org/10.1086/260229.

Decision Theory

Caplin, Dean, & Martin (2011 AER). Search and Satisficing. https://doi.org/10.1257/aer.101.7.2899.

Caplin & Dean (2011 TE). Search, Choice, and Revealed Preference. https://doi.org/10.3982/TE592.

Aguiar, Boccardi, & Dean (2016 JET). Satisficing and stochastic choice. https://doi.org/10.1016/j.jet.2016.08.008.

References II

Consumer Search

Diamon (1971 JET). A model of price adjustment. https://doi.org/10.1016/0022-0531(71)90013-5.

Varian (1980 AER). A Model of Sales. https://doi.org/10.1257/aer.70.4.651.

Wolinsky (1986 QJE). True Monopolistic Competition as a Result of Imperfect Information. https://doi.org/10.2307/1885694/

Stahl (1989 AER). Oligopolistic Pricing with Sequential Consumer Search. https://www.jstor.org/stable/1827927.

Burdett & Judd (1983 Ecta). Equilibrium Price Dispersion. https://doi.org/10.2307/1912661.

Armstrong & Zhou (2016 REstud) Search Deterrence. https://doi.org/10.1093/restud/rdv036.

Hong & Shum (2006 RAND). Using Price Distributions to Estimate Search Costs. https://doi.org/10.1111/j.1756-2171.2006.tb00015.x.

References III

De los Santos, Hortaçsu, & Wildenbeest (2012 AER). Testing Models of Consumer Search Using Data on Web Browsing and Purchasing Behavior. https://doi.org/10.1257/aer.102.6.2955.

Hortaçsu & Syverson (2004 QJE). Product Differentiation, Search Costs, and Competition in the Mutual Fund Industry. https://doi.org/10.1162/0033553041382184

Honka & Chintagunta (2017 MktSc). Simultaneous or Sequential? Search Strategies in the U.S. Auto Insurance Industry. https://doi.org/10.1287/mksc.2016.0995.

References IV

Job Search and Labour Market

McCall (1970 QJE). Economics of Information and Job Search. https://doi.org/10.2307/1879403.

Lucas & Prescott (1974 JET). Equilibrium Search and Unemployment. https://doi.org/10.1016/0022-0531(74)90106-9.

Brown, Flinn, & Schotter (2011 AER). Real-Time Search in the Laboratory and the Market. https://doi.org/10.1257/aer.101.2.948

Kiefer & Neumann (1979 JPE). An Empirical Job-Search Model, with a Test of the Constant Reservation-Wage Hypothesis. https://doi.org/10.1086/260741

Rogerson, Shimer, & Wright (2005 JET). Search-Theoretic Models of the Labor Market: A Survey. https://doi.org/10.1257/002205105775362014.

Chade, Eeckhout, & Smith (2017 JEL). Sorting through Search and Matching Models in Economics. https://doi.org/10.1257/jel.20150777.