Stopping

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Topics in Economic Theory

Overview

1. Why Economic Theory
2. Overview of the Course
3. Stopping: Searching for a Job
4. Optimal Stopping: Existence and Regularity

Overview

Economic Theory

Goals

- Studies behaviour
- Understand how different forces interact and lead to different outcomes
- Positive view: Explain patterns, make predictions
- Normative view: Prescribe behaviour
- Examples: consumer demand and firm pricing, student applications to university, voting, technology adoption, hospital residency program management

(Not particular to theory: in essence, all science strives for generality)

Models as maps, simplified description of reality

- Behavioural implications = Empirical content

This course

- Develop building blocks

Overview of the Course

This term: how people make decisions when faced with uncertainty, limited information, and evolving opportunities

Topics

1. Stopping: accepting a job offer.
2. Stopping and choosing: selling a house.
3. Learning and Choosing: buying a computer.
4. Searching: shopping for clothes
5. Social Learning: checking neighbours' crop yields.
6. Remembering: ordering at a restaurant.
7. Learning in Games: adjusting prices.
8. Common Learning: attacking a currency.

Very unlikely that we'll cover all the topics.

Overview of the Course

Lectures provide fundamentals.

Presentations:

- Every week starting from next week.
- Everyone required to prepare a 15 minute presentation on an assigned paper.
- One person will be selected at random to present.

Overview

1. Why Economic Theory

2. Overview of the Course

3. Stopping: Searching for a Job
   - Job Search
   - Job Search with Discounting

4. Optimal Stopping: Existence and Regularity

Job Search

Accept offer Y_t, continue searching with a per period cost of c.

Interpretation:

- Job search (McCall 1970 QJE): TIOLI salary offers Y_t, cost to search c.
- Selling a house/asset: TIOLI offers Y_t, council tax/management fees c.

$Y_t \sim F$, iid; F continuous, strictly increasing.

Assume $\mathbb{E}[\mathbf{1}_{Y_t \geq 0} Y_t] < \infty$; $Y_0 = 0$; $\mathbb{P}(Y_t > c) > 0$.

Job search

Accept and get Y_t (present value of getting same wage forever);

Refuse and get z and face same problem tomorrow

Markov problem; state variable = Y_t

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Set up Bellman equation; $V(Y_t) = \max\{Y_t, \mathbb{E}[V(Y_{t+1})] - c\}$
(iid $\implies$ stationary problem)

Value: $V(Y_t)$
(handwavy: this presumes a solution and we don't know yet if/why we can do this)

Define $V_t := V(Y_t)$ and $\bar{V} = \mathbb{E}[V(Y_t)]$

Define $\tilde{y} := \bar{V} - c$

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Define $\tilde{y} := \bar{V} - c$

Take expectations and get $\tilde{y} + c = \mathbb{E}[\max\{Y_t, \tilde{y}\}] \iff c = \mathbb{E}[(Y_t - \tilde{y})^+] = \int_{\tilde{y}}^{\infty} y \, \mathrm{d}F(y)$

F continuous and strictly increasing: $\exists! \tilde{y} : c = \mathbb{E}[(Y_t - \tilde{y})^+]$

$\tilde{y}$: reservation value

Optimal rule: continue if and only if $Y_t < \tilde{y}$

Job Search with Discounting

Accept offer Y_t, continue searching and receive z; discount factor $\beta \in (0, 1)$.

Interpretation:

- Job search: TIOLI salary offers Y_t, unemployment subsidy z, cost of time β.
- Selling a house/asset: TIOLI offers Y_t, rent acrued z, interest rate r, discount factor $\beta = (1 + r)^{-1}$.

$Y_t \sim F$, iid; F continuous, strictly increasing.

Assume $\mathbb{E}[1_{Y_t \geq 0} Y_t] < \infty$; $Y_0 = 0$; $\mathbb{P}(Y_t > c) > 0$.

Job Search with Discounting

Define $\hat{Y}_t := \frac{\beta^t}{1-\beta} Y_t$ (present value).

Accept and get Y_t forever $\equiv$ Accept and get $\hat{Y}_t$

Refuse, get z, and face same problem tomorrow but discounted by β.

Markov problem; state variable = $\hat{Y}_t$

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Set up Bellman equation; $V(\hat{Y}_t) = \max\{\hat{Y}_t, z + \beta\mathbb{E}[V(\hat{Y}_{t+1})]\}$

Value: $V(\hat{Y}_t)$

Brief refresher...

Refresher

Definition

$T : X \to X$ is a contraction on (X, d) if $\exists \delta \in [0, 1) : d(T(x), T(y)) \leq \delta d(x, y)\ \forall x, y \in X$.

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Banach Fixed-Point Theorem

Let (X, d) be a non-empty complete metric space and T a contraction mapping on (X, d). Then, $\exists! x^* \in X : T(x^*) = x^*$. Moreover, for any $x_0 \in X$, $x^* = \lim_{n\to\infty} T^n(x_0)$, where $T^{n+1} := T \circ T^n$ and $T^1 := T$.

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Proof

Let $x_n := T^n(x_0)$. Then $d(x_{n+1}, x_n) = d(T^n(x_1), T^n(x_0)) \leq \delta^n d(x_1, x_0)$, hence $\{x_n\}_n$ is a Cauchy sequence.

(X, d) complete $\equiv$ Cauchy sequences converge $\implies$ x_n converges to some $x^* = T(x^*)$.

Take any $y_0 \in X \setminus \{x_0\}$; define $y_n := T^n(y_0)$; $y_n \to y^*$.

If $x^* \neq y^*$, then $d(y^*, x^*) = d(T^n(y^*), T^n(x^*)) = \delta^n d(y^*, x^*) < d(y^*, x^*)$, a contradiction.

Refresher

Blackwell's Conditions for Contraction Mapping

Let $B(X)$ denote the set of bounded real functions on some nonempty set X endowed with the sup-metric d_∞. Suppose $T : B(X) \to B(X)$ satisfies (i) $\forall f, g \in B(X) : f \geq g \implies T(f) \geq T(g)$, and (ii) $\exists \delta \in [0, 1)$ s.t. $T(f + \alpha) \leq T(f) + \delta\alpha \; \forall f \in B(X)$ and $\forall \alpha \in \mathbb{R}_+$. Then T is a contraction.

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Proof

For any $f, g \in B(X)$ and $x \in X$, $f(x) - g(x) \leq |f(x) - g(x)| \leq d_\infty(f,g)$.
(i) and (ii): $f \leq g + d_\infty(f,g) \implies T(f) \leq T(g) + \delta d_\infty(f,g)$
and, symmetrically, $T(g) \leq T(f) + \delta d_\infty(f,g)$.
This implies $d_\infty(T(f), T(g)) \leq \delta d_\infty(f,g)$.

Job Search with Discounting

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Markov problem; state variable = $\hat{Y}_t$

Set up Bellman equation; $V(\hat{Y}_t) = \max\{\hat{Y}_t, z + \beta\mathbb{E}[V(\hat{Y}_{t+1})]\}$

Value: $V(\hat{Y}_t)$, well-defined

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Take expectations and get

$$\bar{V} = \mathbb{E}[\max\{\hat{Y}_t, z + \beta\bar{V}\}] \iff \bar{V}(1-\beta) = z + \mathbb{E}[(\hat{Y}_t - (z + \beta\bar{V}))^+] = \int_{z+\beta\bar{V}}^{\infty} \frac{1}{1-\beta} y \, \mathrm{d}F(y)$$

F continuous: $\exists! \bar{V} : \bar{V}(1-\beta) = z + \mathbb{E}[(\hat{Y}_t - (z + \beta\bar{V}))^+]$

$\tilde{y} := (1-\beta)(z + \beta\bar{V})$: reservation value

Optimal rule: continue if and only if $Y_t < \tilde{y}$

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Going Beyond the Basic Setting

Y_t may not be iid

- Depend on time of unemployment
- Result from underlying dynamic game between recruiting firms
- Uncertain market conditions (hence perception of F evolves over time depending on past Y_ℓ)

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Introduce general tools to tackle the problem

Setup and Assumptions

$\{X_0, X_1, X_2, \ldots\}$ rv whose joint distribution is assumed to be known; write $X^t := (X_\ell)_{\ell=1,\ldots,t}$.

Sequence of functions $x^t \mapsto y_t(x^t) \in \mathbb{R}$; write $Y_t := y_t(x^t)$.

Filtration $\mathbb{F} = \{\mathcal{F}_t\} = \sigma(X^t)$.

Adapted payoff process $\{Y_t\}$; terminal Y_∞ (possibly $-\infty$).

Stopping time τ: $\{\tau \leq t\} \in \mathcal{F}_t$ for all t; feasible set $\mathbb{T}$.

Objective: maximise value of Y by adequately choosing stopping time, $\sup_{\tau \in \mathbb{T}} \mathbb{E}[Y_\tau]$.

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Two questions:

1. When is there actually an optimal stopping time? (Is sup actually a max?)
2. If so, what does it look like?

   Previous applications: guess and verify or use specific structural assumptions.
   Now: use very general assumptions.

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Standing assumptions

(A1) $\mathbb{E}\left[\sup_{t \geq 0} Y_t\right] < \infty$.

(A2) $\lim_{t \to \infty} \mathbb{E}[Y_t] \leq Y_\infty$ a.s.

Note: (A1) implies $\sup_\tau \mathbb{E}[Y_\tau] < \infty$

Regular Stopping Times

Definition (Regularity)

τ is regular if for all t, $\mathbb{E}[Y_\tau \mid \mathcal{F}_t] > Y_t$ a.s. on $\{\tau > t\}$.

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Under (A1), for any stopping time τ there exists a *regular* stopping time $\rho \leq \tau$ with $\mathbb{E}[Y_\rho] \geq \mathbb{E}[Y_\tau]$.

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Under (A1), for any stopping time τ there exists a *regular* stopping time $\rho \leq \tau$ with $\mathbb{E}[Y_\rho] \geq \mathbb{E}[Y_\tau]$.

Lemma 2 (Regularity is closed under $\vee$)

Under (A1), if τ and ρ are regular, then $\xi := \tau \vee \rho$ is regular and $\mathbb{E}[Y_\xi] \geq \max\{\mathbb{E}[Y_\tau], \mathbb{E}[Y_\rho]\}$.

Proof of Lemma 1 (Regularity wloo)

Proof

Fix τ with $\mathbb{E}[|Y_\tau|] < \infty$ (true by (A1) since $Y_\tau \leq \sup_s Y_s$).

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Define $Z_t := \mathbb{E}[Y_\tau \mid \mathcal{F}_t]$ and let $\rho := \inf\{t \geq \mathbf{0} : Z_t \leq Y_t\}$.

On $\{\rho > t\}$: $Y_t < Z_t = \mathbb{E}[Y_\tau \mid \mathcal{F}_t]$, so ρ is regular.

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On $\{\rho = t\}$: $Y_\rho = Y_t \geq Z_t = \mathbb{E}[Y_\tau \mid \mathcal{F}_t]$. On $\{\rho = \infty\}$: $Y_\rho = Y_\infty = Y_\tau$ a.s.

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Hence

$$\mathbb{E}[Y_\rho] = \sum_{t=0}^{\infty} \mathbb{E}[\mathbf{1}_{\{\rho=t\}} Y_t] + \mathbb{E}[\mathbf{1}_{\{\rho=\infty\}} Y_\infty]$$

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On $\{\rho = t\}$: $Y_\rho = Y_t \ge Z_t = \mathbb{E}[Y_\tau \mid \mathcal{F}_t]$. On $\{\rho = \infty\}$: $Y_\rho = Y_\infty = Y_\tau$ a.s.

Hence

$$\begin{aligned}\mathbb{E}[Y_\rho] &= \sum_{t=\mathbf{0}}^{\infty} \mathbb{E}[\mathbf{1}_{\{\rho=t\}} Y_t] + \mathbb{E}[\mathbf{1}_{\{\rho=\infty\}} Y_\infty] \\ &\ge \sum_{t=\mathbf{0}}^{\infty} \mathbb{E}[\mathbf{1}_{\{\rho=t\}} \mathbb{E}[Y_\tau \mid \mathcal{F}_t]] + \mathbb{E}[\mathbf{1}_{\{\rho=\infty\}} Y_\tau]\end{aligned}$$

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Hence

$$\begin{aligned}
\mathbb{E}[Y_\rho] &= \sum_{t=0}^{\infty} \mathbb{E}[\mathbf{1}_{\{\rho=t\}} Y_t] + \mathbb{E}[\mathbf{1}_{\{\rho=\infty\}} Y_\infty] \\
&\geq \sum_{t=0}^{\infty} \mathbb{E}[\mathbf{1}_{\{\rho=t\}} \mathbb{E}[Y_\tau \mid \mathcal{F}_t]] + \mathbb{E}[\mathbf{1}_{\{\rho=\infty\}} Y_\tau] \\
&\geq \sum_{t=0}^{\infty} \mathbb{E}[\mathbf{1}_{\{\rho=t\}} Y_\tau] + \mathbb{E}[\mathbf{1}_{\{\rho=\infty\}} Y_\tau] \\
&= \mathbb{E}[Y_\tau].
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On $\{\rho > t\}$: $Y_t < Z_t = \mathbb{E}[Y_\tau \mid \mathcal{F}_t]$, so ρ is regular.

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&= \mathbb{E}[Y_\tau].
\end{aligned}$$

Suppose $\neg(\rho \leq \tau)$; note that, at $\{\rho > \tau = t\}$, $Z_t = Z_\tau = Y_\tau < Z_t$, a contradiction.

Proof of Lemma 2 (Regularity is closed under ∨)

Proof

1. Proving ξ is regular:

   $\{\xi > t\} = \{\xi = \tau > t\} \cup \{\xi = \rho > t\}$.

Proof of Lemma 2 (Regularity is closed under $\vee$)

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1. Proving ξ is regular:

   $\{\xi > t\} = \{\xi = \tau > t\} \cup \{\xi = \rho > t\}$.

   On $\{\xi = \tau > t\}$, $\mathbb{E}[Y_\xi \mid \mathcal{F}_t] = \mathbb{E}[Y_\tau \mid \mathcal{F}_t] > Y_t$ a.s. $\because$ τ is regular.

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   Symmetrically, on $\{\xi = \rho > t\}$, $\mathbb{E}[Y_\xi \mid \mathcal{F}_t] = \mathbb{E}[Y_\rho \mid \mathcal{F}_t] > Y_t$ a.s. $\because$ ρ is regular.

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2. Proving $\mathbb{E}[Y_\xi] \geq \mathbb{E}[Y_\tau] \vee \mathbb{E}[Y_\rho]$:

   On $\{\xi = \tau = t\}$, $Y_\xi = Y_\tau = Y_t$.

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   On $\{\xi = \tau = t\}$, $Y_\xi = Y_\tau = Y_t$.

   On $\{\xi = \rho > \tau = t\}$, $\xi = \rho$ and $\mathbb{E}[Y_\xi \mid \mathcal{F}_t] = \mathbb{E}[Y_\rho \mid \mathcal{F}_t] > Y_t = Y_\tau$ a.s.

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   Hence

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\mathbb{E}[Y_\xi] &= \sum_{t=0}^{\infty} \mathbb{E}[\mathbf{1}_{\{\tau=t\}} Y_\xi] + \mathbb{E}[\mathbf{1}_{\{\tau=\infty\}} Y_\xi] = \sum_{t=0}^{\infty} \mathbb{E}[\mathbf{1}_{\{\tau=t\}} \mathbb{E}[Y_\xi \mid \mathcal{F}_t]] + \mathbb{E}[\mathbf{1}_{\{\tau=\infty\}} Y_\xi] \\
&\geq \sum_{t=0}^{\infty} \mathbb{E}[\mathbf{1}_{\{\tau=t\}} Y_\tau] + \mathbb{E}[\mathbf{1}_{\{\tau=\infty\}} Y_\tau] = \mathbb{E}[Y_\tau].
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   $\{\xi > t\} = \{\xi = \tau > t\} \cup \{\xi = \rho > t\}$.

   On $\{\xi = \tau > t\}$, $\mathbb{E}[Y_\xi \mid \mathcal{F}_t] = \mathbb{E}[Y_\tau \mid \mathcal{F}_t] > Y_t$ a.s. $\because$ τ is regular.

   Symmetrically, on $\{\xi = \rho > t\}$, $\mathbb{E}[Y_\xi \mid \mathcal{F}_t] = \mathbb{E}[Y_\rho \mid \mathcal{F}_t] > Y_t$ a.s. $\because$ ρ is regular.

2. Proving $\mathbb{E}[Y_\xi] \geq \mathbb{E}[Y_\tau] \vee \mathbb{E}[Y_\rho]$:

   On $\{\xi = \tau = t\}$, $Y_\xi = Y_\tau = Y_t$.

   On $\{\xi = \rho > \tau = t\}$, $\xi = \rho$ and $\mathbb{E}[Y_\xi \mid \mathcal{F}_t] = \mathbb{E}[Y_\rho \mid \mathcal{F}_t] > Y_t = Y_\tau$ a.s.

   Hence

$$\mathbb{E}[Y_\xi] = \sum_{t=0}^{\infty} \mathbb{E}[\mathbf{1}_{\{\tau=t\}} Y_\xi] + \mathbb{E}[\mathbf{1}_{\{\tau=\infty\}} Y_\xi] = \sum_{t=0}^{\infty} \mathbb{E}[\mathbf{1}_{\{\tau=t\}} \mathbb{E}[Y_\xi \mid \mathcal{F}_t]] + \mathbb{E}[\mathbf{1}_{\{\tau=\infty\}} Y_\xi]$$

$$\geq \sum_{t=0}^{\infty} \mathbb{E}[\mathbf{1}_{\{\tau=t\}} Y_\tau] + \mathbb{E}[\mathbf{1}_{\{\tau=\infty\}} Y_\tau] = \mathbb{E}[Y_\tau].$$

   By a symmetric argument, $\mathbb{E}[Y_\xi] \geq \max\{\mathbb{E}[Y_\tau], \mathbb{E}[Y_\rho]\}$.

Existence

Theorem (Existence)

Under (A1) and (A2), there is a regular τ such that $\mathbb{E}[Y_\tau] = \sup_{\rho\in\mathbb{T}} \mathbb{E}[Y_\rho]$.

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Take the case $V^* := \sup_{\rho\in\mathbb{T}} \mathbb{E}[Y_\rho] > -\infty$.

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Define $\tau_\infty := \sup_{n \in \mathbb{N}} \rho_n$. τ_n pointwise increasing $\implies$ pointwise converges to τ_∞.
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Conclude: $V^* = \sup_{\rho \in \mathbb{T}} \mathbb{E}[Y_\rho] \geq \mathbb{E}[Y_{\tau_\infty}] \geq V^*$.

Assumptions

Example

Let $X_t \sim$Bernoulli(1/2) iid; $Y_0 := \mathbf{0}$, $Y_t := (\mathbf{2}^t - \mathbf{1}) \prod_{\ell=\mathbf{1}}^{t} X_\ell$ for $t \in \mathbb{N}$, $Y_\infty := \mathbf{0}$.

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Fails (A1): Note $\sup_{k \le t} Y_k = 2^k - 1$ with probability $2^{-(k+1)}$ for $k = 0, 1, \ldots, t-1$ and with probability 2^{-t} for $k = t$. Hence $\mathbb{E}[\sup_t Y_t] = \sum_{t=0}^{\infty}(2^t - 1)2^{-(t+1)} = \infty$.

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Indeed, no optimal stopping time. Conditional on reaching t with $Y_t > 0 \iff \prod_{\ell=1}^t X_\ell = 1$, then don't want to stop: $Y_t = 2^t - 1 < 2^t - 1/2 = (1/2)(2^{t+1} - 1) = \mathbb{E}[Y_{t+1} | Y_t > 0]$.

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Indeed, no optimal stopping time as $Y_t < Y_{t+1}$.

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Want something like Bellman equation/DPP: stop today or continue assuming optimal stopping from then on

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Definition

Let $(X_t)_{t\in T}$ be a collection of rv. Z rv is essential supremum of $(X_t)_{t\in T}$, $Z = \operatorname{ess\,sup}_{t\in T} X_t$, if (i) $\mathbb{P}(Z \geq X_t) = 1\ \forall t \in T$ ('probabilistic upper bound'), and (ii) $\forall Z' : \mathbb{P}(Z' \geq X_t) = 1\ \forall t \in T$, $\mathbb{P}(Z' \geq Z) = 1$ (smallest probabilistic upper bound).

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Let $(X_t)_{t \in T}$ be any collection of rv.
An essential supremum always exists.
Furthermore, $\exists$ a countable $C \subset T : \sup_{t \in C} X_t = \operatorname{ess\,sup}_{t \in T} X_t$.

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Let $U \sim U(0,1)$, $T = [0,1]$, and $X_t = 1_{\{c=t\}}$. $\sup_{t \in T} X_t = 1 \neq \operatorname{ess\,sup}_{t \in T} X_t = 0$.

Regularity from T Onward

Notation:

"$X \geq Y$" $\equiv \mathbb{P}(X \geq Y) = 1$.

"$X \geq Y$ on A" $\equiv \mathbb{P}(\{X \geq Y\} \cap A) = \mathbb{P}(A)$.

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Lemma 2' (Regularity is closed under $\vee$)

Under (A1), if $\tau \geq T$ and $\rho \geq T$ are regular from T onward, then $\xi := \tau \vee \rho$ is regular from T onward and $\mathbb{E}[Y_\xi] \geq \max\{\mathbb{E}[Y_\tau], \mathbb{E}[Y_\rho]\}$.

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$$V_t^* := \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}[Y_\tau \mid \mathcal{F}_t]$$

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   By the lemmas 1' and 2',

$$V_t^* \geq \mathbb{E}[Y_{\tau_n} \mid \mathcal{F}_t] = \mathbb{E}[\mathbb{E}[Y_{\tau_n} \mid \mathcal{F}_{t+1}] \mid \mathcal{F}_t] \geq \mathbb{E}\left[\max_{\ell=1,\ldots,n} \mathbb{E}[Y_{\hat{\tau}_\ell} \mid \mathcal{F}_{t+1}] \middle| \mathcal{F}_t\right] \to \mathbb{E}[V_{t+1}^* \mid \mathcal{F}_t]$$

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   Define regularised from $t+1$ onward $\rho_n := \inf\{\ell > t \mid \mathbb{E}[Y_{\hat{\tau}_n}] \leq Y_\ell\}$; let $\tau_n := \max\{\rho_1, \rho_2, \ldots, \rho_n\}$, regular from t onward.

   By the lemmas 1' and 2',

$$V_t^* \geq \mathbb{E}[Y_{\tau_n} \mid \mathcal{F}_t] = \mathbb{E}[\mathbb{E}[Y_{\tau_n} \mid \mathcal{F}_{t+1}] \mid \mathcal{F}_t] \geq \mathbb{E}\left[\max_{\ell=1,\ldots,n} \mathbb{E}[Y_{\hat{\tau}_\ell} \mid \mathcal{F}_{t+1}] \middle| \mathcal{F}_t\right] \to \mathbb{E}[V_{t+1}^* \mid \mathcal{F}_t]$$

   Since, trivially, $V_t^* \geq Y_t$, we get $V_t^* \geq \max\{Y_t, \mathbb{E}[V_{t+1}^* \mid \mathcal{F}_t]\}$.

Dynamic Programming Principle

$$V_t^* := \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}[Y_\tau \mid \mathcal{F}_t] = \max\{Y_t, \mathbb{E}[V_{t+1}^* \mid \mathcal{F}_t]\}; \qquad \tau^* := \inf\{t \geq 0 \mid Y_t = V_t^*\}$$

Example

Let $Y_0 := 0$, $Y_t := 1 - 1/t$ for $t \in \mathbb{N}$, $Y_\infty := 0$.

Satisfies (A1): $Y_t \leq 1$.

Fails (A2): $Y_t \to 1 > 0 = Y_\infty$.

Indeed, no optimal stopping time as $Y_t < Y_{t+1}$.

Note: $\tau^* = \infty$ and $Y_{\tau^*} = 0 < V_t^*$.

Characterising Optimal Stopping Time

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Lemma

Take any stopping time τ. Under (A1), $\mathbb{E}[Y_{\tau \wedge \tau^*}] \geq \mathbb{E}[Y_\tau]$.

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Stopping whenever τ^* says to stop can only improve the expected payoff.

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&\geq \mathbb{E}\left[\sum_{t=0}^{\infty} \mathbf{1}_{\{\tau^* = t < \tau\}} \mathbb{E}[Y_\tau \mid \mathcal{F}_t]\right] + \mathbb{E}[\mathbf{1}_{\{\tau^* \geq \tau\}} Y_\tau] = \mathbb{E}[Y_\tau].
\end{aligned}$$

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Theorem (Optimal Stopping Time)

Under (A1), if an optimal stopping time exists, τ^* is optimal.

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Let τ be an optimal stopping time.

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Let τ be an optimal stopping time.

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Finally, by Lemma 2, $\tau'' \vee \tau^*$ must also be optimal. Note that $\tau'' \vee \tau^* = \tau^*$ by construction.

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It can be shown that τ^* is the **earliest optimal stopping time**, i.e., $\tau^* \leq \tau \; \forall$ optimal τ.
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Another stopping time: $\tau^{**} := \inf\{t \geq 0 \mid Y_t > \mathbb{E}[V_{t+1}^* \mid \mathcal{F}_t]\}$

It can be shown that τ^{**} is the **latest optimal stopping time**, i.e., $\tau \leq \tau^{**} \; \forall$ optimal τ.