Conflict and Volatility

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Topics in Economic Theory

Overview

1. Conflict and Volatility
   - Choosing Instability
   - A Detour Through Viscosity Solutions
   - Back to Choosing Instability
   - Equilibria
   - Discussion

Instability

Social Instability:

Groups create instability to induce social change:

demonstrations, protests, riots, civil disobedience, etc.

Groups with opposing interests use instability to advance their agenda.

Uncertainty and Instability:

Instability generates uncertainty; and often it's all it does.

Actions that have no predictable effect other than increasing instability.

Equally likely to advance interests as to backfire.

Timing and Environment (Barilla & Gonçalves (2024 TE))

Time is continuous $t \in \mathbb{R}_+$.

$X_t \in [\mathbf{0}, \mathbf{1}]$: state at time t.

A and B: opposing preferences over status quo; constant-sum gross payoffs $X_t, \mathbf{1} - X_t$.

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Dynamics:

$$dX_t = \sqrt{2(\alpha_t + \beta_t)}dB_t - dK_t$$

B_t: standard Brownian Motion.

Endogenous Instability: $\alpha_t + \beta_t$ scale instantaneous volatility of Brownian motion.

$\alpha_t \geq 0$ (resp. $\beta_t \geq 0$) controlled by player A (resp. B).

Remain at X_t if both players choose not to trigger instability $\alpha_t = \beta_t = 0$.

K_t: regulator process keeping X_t inside $[0,1]$; $X_t \in (0,1) \Rightarrow dK_t = 0$.

Instability has symmetric effect everywhere but at boundary.

Payoffs and Equilibrium

Instantaneous (net) payoffs:

$$u_a(X_t, \alpha_t) := X_t - \frac{c_a}{2}\alpha_t^2, \qquad u_b(X_t, \beta_t) := (1 - X_t) - \frac{c_b}{2}\beta_t^2.$$

Generating instability is costly: focus on quadratic cost (relaxed).

Not zero-sum: creating instability requires pure destruction of surplus.

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Equilibrium concept:

Players maximise average discounted payoffs.

Markov-Perfect Equilibria in continuous strategies: $\alpha_t = a(X_t)$, $\beta_t = b(X_t)$.

Best Response: Control Problem

Problem is symmetric; wlog focus on player A's problem.

Fix arbitrary continuous $b : [0,1] \to \mathbb{R}_+$ (player B's strategy).

Player A's control problem (best response):

$$v_a(x) := \sup_{\alpha} \mathbb{E}\left[\int_0^{\infty} r_a e^{-r_a t}\left(X_t - \frac{c_a}{2}\alpha_t^2\right) dt\right]$$

$$dX_t = \sqrt{2\left(\alpha_t + b(X_t)\right)}dB_t - dK_t.$$

Differential characterisation: HJB + boundary conditions (due to $X_t \in [0,1]$)

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1. Conflict and Volatility
   - Choosing Instability
   - A Detour Through Viscosity Solutions
     - The Problem
     - Sub/Supersolutions, Test Functions, and Super/subjets
     - Viscosity Solutions
     - Optimal Control and Viscosity Solutions
   - Back to Choosing Instability
   - Equilibria
   - Discussion

The Viscosity Setup

The reference: Crandall, Ishii, & Lions (1992 BullAMS).

Problem: solve fully nonlinear 2nd order partial differential equations (read: 'complicated HJB').

- 2nd order PDE: $F(x, u(x), \mathrm{d}u(x), \mathrm{d}^2u(x)) = \mathbf{0}$ for $x \in D$

  $F : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n \to \mathbb{R}$,

  $x \in \mathbb{R}^n$: argument.

  $u(x) \in \mathbb{R}$: value of the function at x.

  $\mathrm{d}u(x) \in \mathbb{R}^n$: gradient of u at x.

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- F is **proper**: $F(x, r, p, M) \leq F(x, s, p, M)$ if $r \leq s$.

Classical Solutions

Classical **solution** $u \in \mathcal{C}^2$: $F(x, u(x), \mathrm{d}u(x), \mathrm{d}^2u(x)) = \mathbf{0}$ for all x.

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Classical **sub**solution $u \in \mathcal{C}^2 : F(x, u(x), \mathrm{d}u(x), \mathrm{d}^2u(x)) \leq 0$ for all x.

Classical **super**solution $u \in \mathcal{C}^2 : F(x, u(x), \mathrm{d}u(x), \mathrm{d}^2u(x)) \geq 0$ for all x.

Superjets and Test Functions

Classical **sub**solution $u \in \mathcal{C}^2 : F(x, u(x), \mathrm{d}u(x), \mathrm{d}^2u(x)) \leq 0$ for all x.

Subsolutions and Test Functions

If u is **sub**solution, $\varphi \in \mathcal{C}^2$, $u \leq \varphi$ locally around x_0, and x_0 is a local **max** of $u - \varphi$, then

(FOC) $\mathrm{d}(u - \varphi)(x_0) = 0 \iff \mathrm{d}u(x_0) = \mathrm{d}\varphi(x_0)$ and

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$\implies F(x_0, u(x_0), \mathrm{d}\varphi(x_0), \mathrm{d}^2\varphi(x_0)) \leq F(x_0, u(x_0), \mathrm{d}u(x_0), \mathrm{d}^2u(x_0)) \leq 0.$

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Local Approximations

As x_0 is local **max** of $u - \varphi$, for $x \to x_0$,

$$u(x) - \varphi(x) \leq u(x_0) - \varphi(x_0)$$

$$\iff u(x) \leq u(x_0) - \varphi(x_0) + \varphi(x)$$

$$= u(x_0) + \mathrm{d}\varphi(x_0)(x - x_0) + \frac{1}{2}(x - x_0)^T \mathrm{d}^2\varphi(x_0)(x - x_0) + o(|x - x_0|^2)$$

where we 2nd order Taylor expanded φ around x_0.

Superjets and Test Functions

Superjet

For $u : D \to \mathbb{R}$, $x_0 \in D$, the **superjet** of u at x_0

$$J_D^{2,+}u(x_0) := \left\{ (p, M) \in \mathbb{R}^n \times \mathcal{S}^n : \begin{array}{c} u(x) \leq u(x_0) + p(x - x_0) + \frac{1}{2}(x - x_0)^T M(x - x_0) \\ +o(|x - x_0|^2) \text{ as } x \to x_0 \end{array} \right\}.$$

Equivalent to set of test functions:

$$J_D^{2,+}u(x_0) \equiv \left\{ (d\varphi(x_0), d^2\varphi(x_0)) : \varphi \in C^2 \text{ and } u - \varphi \text{ has a local } \textbf{max} \text{ at } x_0 \right\}.$$

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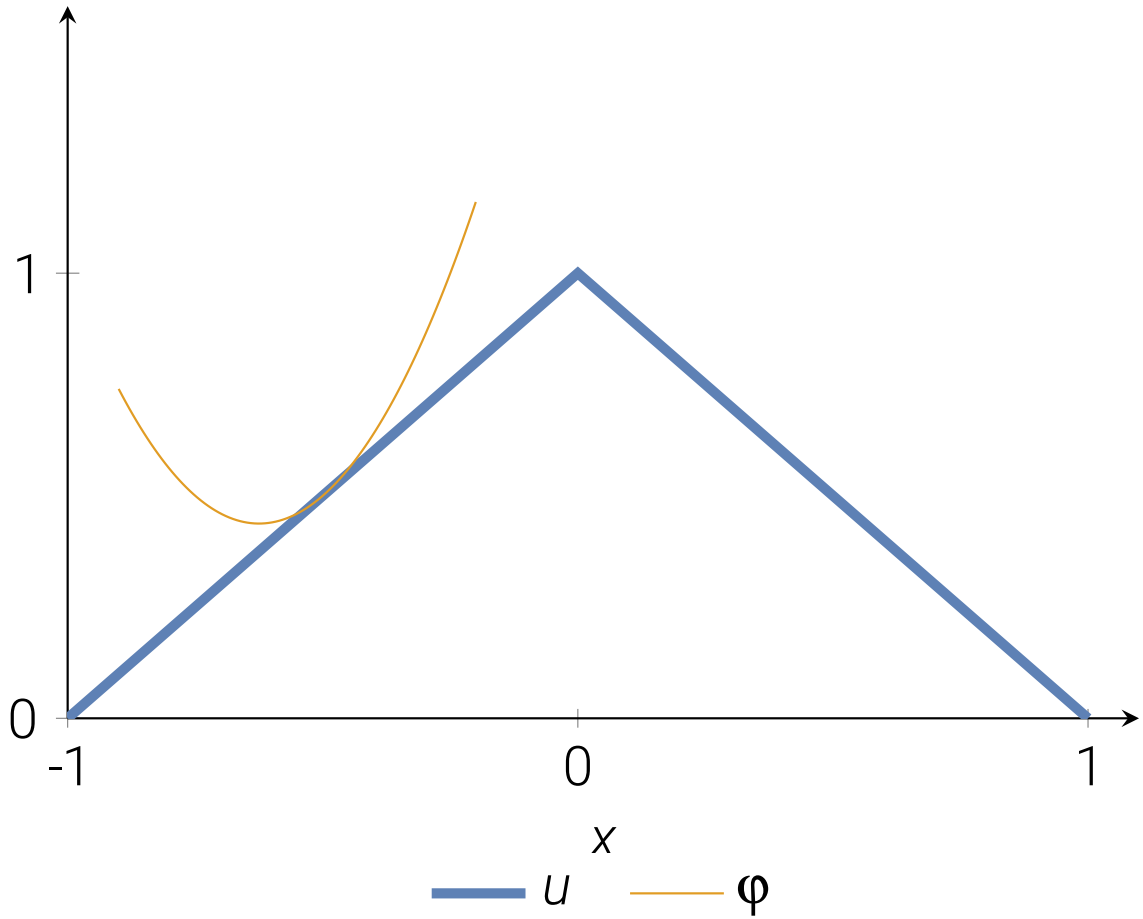
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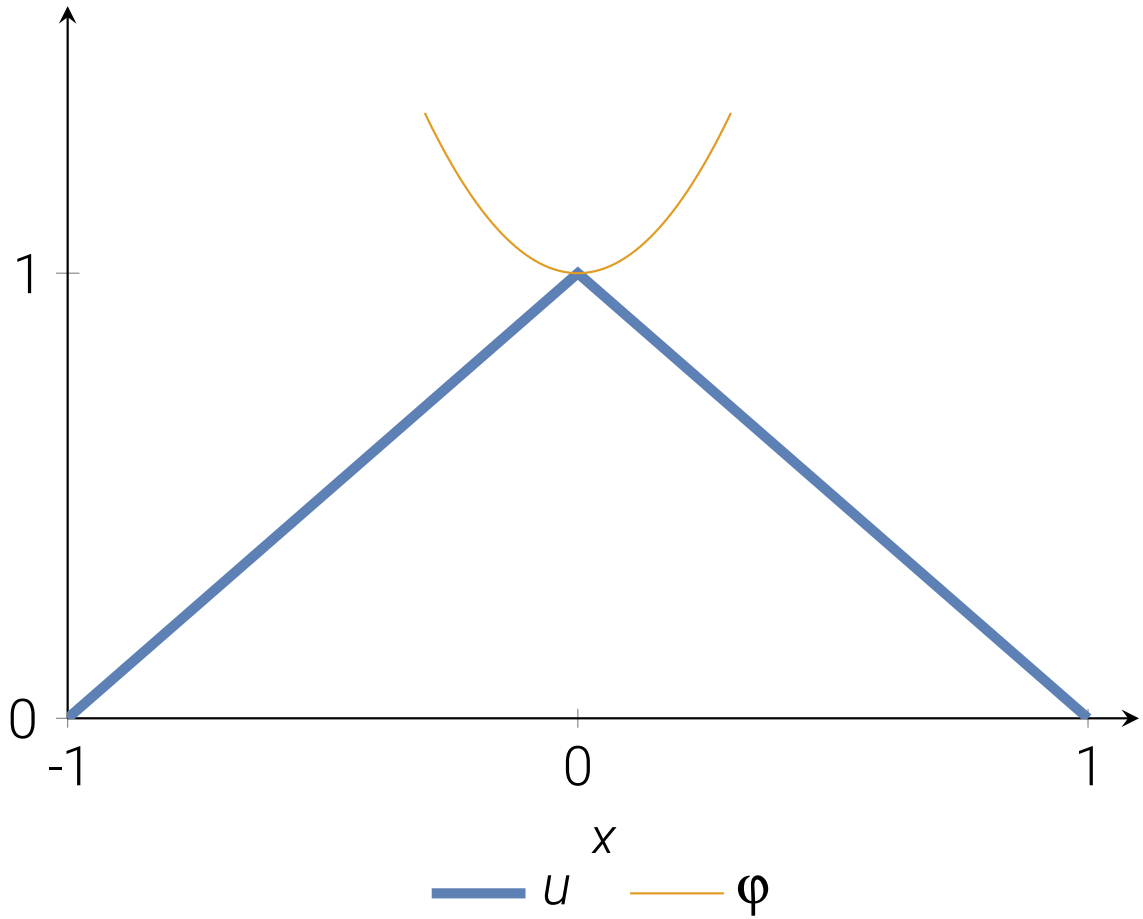
Closure of Superjets:

$$\overline{J}_D^{2,+}u(x_0) := \left\{ (p, M) \in \mathbb{R}^n \times \mathcal{S}^n : \begin{array}{l} \exists (x_n, p_n, M_n) \in D \times \mathbb{R}^n \times \mathcal{S}^n \text{ such that} \\ (p_n, M_n) \in J_D^{2,+}u(x_n) \ \forall n \\ (x_n, u(x_n), p_n, M_n) \to (x_0, u(x_0), p, M) \end{array} \right\}$$

Supergradients and Superjets



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Superjets and Supergradients

Supergradients capture local first-order information about the function.

Superjets capture local second-order information about the function.

$p \in \bar{\partial} u(x_0) \implies (p, 0) \in J^{2,+} u(x_0) \implies (p, M) \in J^{2,+} u(x_0), \ \forall M \geq 0.$

p is in supergradient of u at x_0, then $(p, 0)$ is in the superjet.

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- Super/Subjets of differences: For any $\varphi \in \mathcal{C}^2$, $J_D^{2,+}(u - \varphi)(x) = \{(p - \mathrm{d}\varphi, M - \mathrm{d}^2\varphi) : (p, M) \in J_D^{2,+}u(x)\}$.

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- Properties of Super/Subjets: $J_D^{2,+}u(x)$ and $J_D^{2,-}u(x)$ are convex.

  $\{M \mid (p, M) \in J_D^{2,+}u(x)\}$ and $\{M \mid (p, M) \in J_D^{2,-}u(x)\}$ are closed; ($J_D^{2,+}u(x)$ and $J_D^{2,-}u(x)$ are not necessarily closed).
- Super/Subjets of differences: For any $\varphi \in \mathcal{C}^2$, $J_D^{2,+}(u - \varphi)(x) = \{(p - \mathrm{d}\varphi, M - \mathrm{d}^2\varphi) : (p, M) \in J_D^{2,+}u(x)\}$.
- Behaviour at the boundary of the domain: $\overline{J}_D^{2,+}u(x)$ can be particular at boundary ∂D.

Superjets, Subjets, and Test Functions

Behaviour at the boundary of the domain:

- Consider $u(x) := 1 - |x|$ and the test function $\varphi(x) := 1 + (1 + 2\varepsilon)x - \frac{1}{2}x^2$, $\varepsilon > 0$.

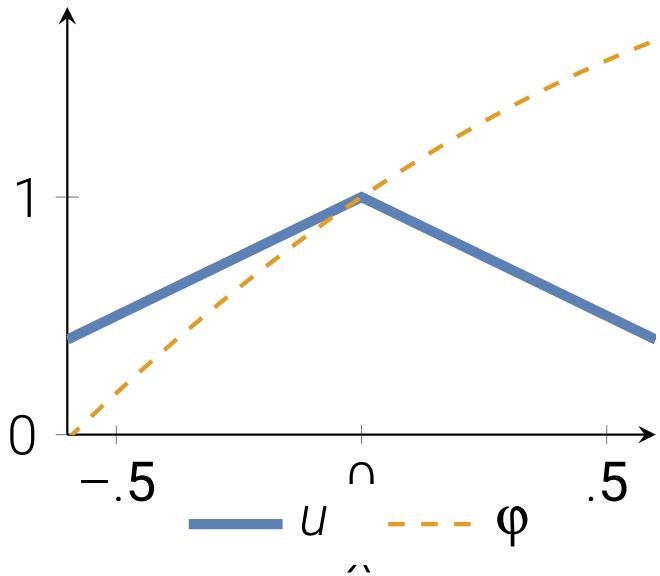


Figure: $\varepsilon = 0.2$.

- On $D = [0, 1]$: For $x \geq 0$,

$$\varphi(x) - u(x) = (2 + 2\varepsilon)x - \frac{1}{2}x^2 \geq 0$$

locally around 0. Hence

$$(1 + 2\varepsilon, -1) \in J^{2,+}_{[0,1]}u(0).$$

$$\Longrightarrow \quad (1 + 2\varepsilon, -1) \in J^{2,+}_{[0,1]}u(0), \quad \text{but} \quad (1 + 2\varepsilon, -1) \notin \overline{J}^{2,+}_{[-1,1]}u(0).$$

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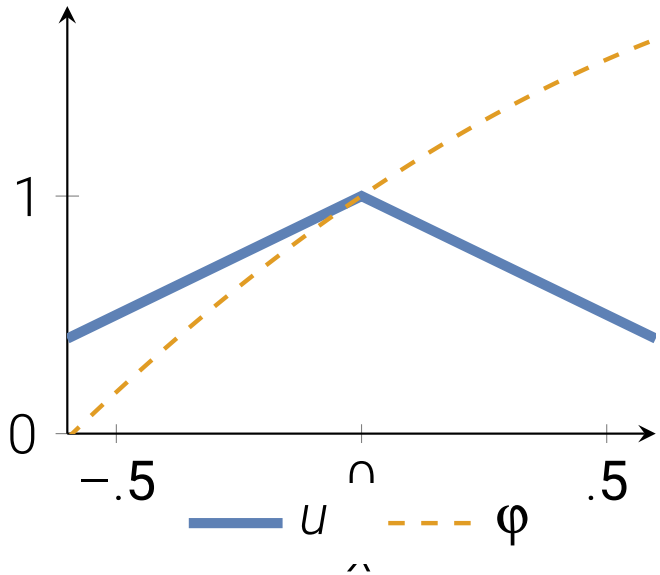


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Thus φ does not touch u from above.

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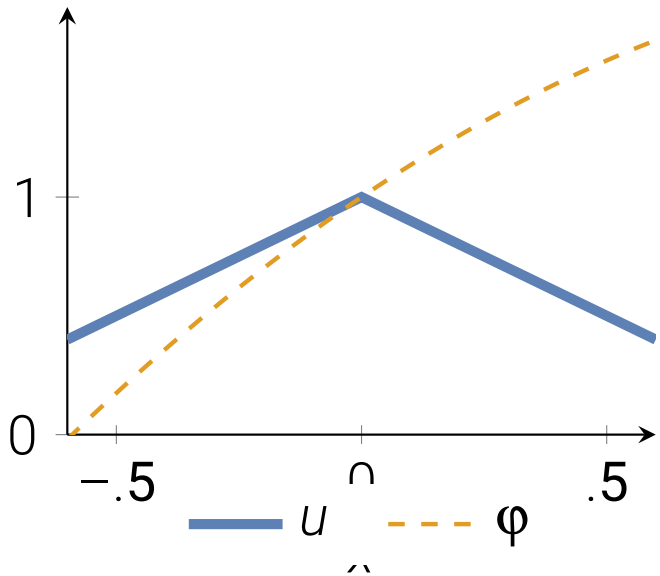


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Thus φ does not touch u from above.

- Even for the closure: $u'(x) = \pm 1$ for $x \neq 0$, and superjet slopes at 0 lie in $[-1, 1]$. Thus

$$(p, M) \in \overline{J}^{2,+}_{[-1,1]}u(0) \implies p \in [-1, 1].$$

$$\implies (1 + 2\varepsilon, -1) \in J^{2,+}_{[0,1]}u(0), \quad \text{but} \quad (1 + 2\varepsilon, -1) \notin \overline{J}^{2,+}_{[-1,1]}u(0).$$

Superjets, Subjets, and Test Functions

Toward bypassing differentiability:

If u is classical **sub**soln, $F(x, u(x), \mathrm{d}u(x), \mathrm{d}^2u(x)) \leq 0$ for $x \in D$, then for all $(p, M) \in J_D^{2,+}u(x)$, $F(x, u(x), p, M) \leq 0$.

Same holds for $(p, M) \in \overline{J}_D^{2,+}u(x)$, with continuous F.

If u is classical **super**soln, $F(x, u(x), \mathrm{d}u(x), \mathrm{d}^2u(x)) \geq 0$ for $x \in D$, then for all $(p, M) \in J_D^{2,-}u(x)$, $F(x, u(x), p, M) \geq 0$.

Same holds for $(p, M) \in \overline{J}_D^{2,-}u(x)$, with continuous F.

Viscosity Solutions

Viscosity Solutions

Let F be proper.

(i) $u \in \text{USC}(D)$ is a **viscosity subsolution** of $F = 0$ on D if $F(x, u(x), p, M) \leq 0$ for all $x \in D$ and $(p, M) \in J_D^{2,+} u(x)$.

(ii) $u \in \text{LSC}(D)$ is a **viscosity supersolution** of $F = 0$ on D if $F(x, u(x), p, M) \geq 0$ for all $x \in D$ and $(p, M) \in J_D^{2,-} u(x)$.

(iii) u is a **viscosity solution** of $F = 0$ on D if it is both a viscosity sub- and supersolution.

If F is continuous, then can use $\overline{J}^{2,+} u, \overline{J}^{2,-} u$.

A classical solution is a viscosity solution.

Viscosity Solutions for Boundary Value Problems

Boundary Value Problem (BVP) on $D \subset \mathbb{R}^n$ (open, bounded):

$F(x, u(x), \mathrm{d}u(x), \mathrm{d}^2u(x)) = \mathbf{0}$ in D.

$B(x, u(x), \mathrm{d}u(x), \mathrm{d}^2u(x)) = \mathbf{0}$ on ∂D.

Examples:

Dirichlet Problem: $B(x, r, p, M) = r - f(x)$, $f \in \mathcal{C}(\partial D)$

E.g. optimal stopping.

Neumann Problem: $B(x, r, p, M) = n(x)p + f(x, r)$, $f \in \mathcal{C}(\partial D \times \mathbb{R})$

where $n(x)$ outer normal unit vector (e.g. $D = (\mathbf{0}, \mathbf{1})$, then $n(\mathbf{1}) = \mathbf{1}$, $n(\mathbf{0}) = -\mathbf{1}$).

This will be the relevant one for our case.

Viscosity Solutions of (BVP)

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Let F, B be proper.

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$F(x,u(x),p,M) \leq \mathbf{0}$ for all $x \in D$ and $(p,M) \in \overline{J}_D^{2,+} u(x)$, and

$F(x,u(x),p,M) \wedge B(x,u(x),p,M) \leq \mathbf{0}$ for all $x \in \partial D$ and $(p,M) \in \overline{J}_{\overline{D}}^{2,+} u(x)$.

(ii) $u \in \mathrm{LSC}(\overline{D})$ is a **viscosity supersolution** (BVP) if

$F(x,u(x),p,M) \geq \mathbf{0}$ for all $x \in D$ and $(p,M) \in \overline{J}_D^{2,-} u(x)$, and

$F(x,u(x),p,M) \vee B(x,u(x),p,M) \geq \mathbf{0}$ for all $x \in \partial D$ and $(p,M) \in \overline{J}_{\overline{D}}^{2,-} u(x)$.

(iii) u is a **viscosity solution** of (BVP) if it is both a viscosity sub- and supersolution.

Viscosity solns also relax BC for sub/supersolutions.

Optimal Control

Setup

Control: $\alpha = \{a_t\}_t \in \mathbb{A}$, taking values in $\mathcal{A}$, compact.

Ito Process: $dX_t = \mu(X_t, a_t)\,dt + \sigma(X_t, a_t)\,dB_t$

μ, σ Lipschitz continuous in x, uniformly in a.

Stopping: $\tau := \inf\{t \geq 0 \mid X_t \notin D\}$

D is open and, for simplicity, convex and bounded.

Payoff: $V(x, \alpha) = \mathbb{E}\left[\int_0^\tau \exp(-rt) f(X_t, a_t)\,dt + \exp(-r\tau) g(X_\tau)\right]$.

Flow payoff and Scrap value: f, g continuous and bounded (can be generalised).

Optimal Control Problem (OC): $V(x) := \sup_{\alpha \in \mathbb{A}} V(x, \alpha)$.

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Boundary Value Problem (BVP)

HJB: for $x \in D$,

$$F(x, V(x), V'(x), V''(x)) := -\sup_{a \in \mathcal{A}} \left\{ f(x,a) + ((\mathcal{L}^a - r)V)(x) \right\}$$

$$= -\sup_{a \in \mathcal{A}} \left\{ f(x,a) + \mu(x,a) V_x(x,a) + \frac{1}{2}\sigma(x,a)^2 V_{xx}(x) - rV(x) \right\} = 0.$$

BC: for $x \in \partial D$, $\quad B(x, V(x), V'(x), V''(x)) := V(x) - g(x) = 0$.

Optimality

Theorem (Lions (1983) Part 2; Fleming & Soner (2006), Ch. V.3)

If V given by (OC) is continuous, then V is a viscosity solution of (BVP).

If $\tilde{V} \in \mathcal{C}_b(\overline{D})$ is a viscosity solution of (BVP), then $\tilde{V} = V$ given by (OC).

Comparison Principle

Theorem (7.9 CIL 92)

Let $D \subset \mathbb{R}^n$ be open, convex, and bounded and F be proper degenerate elliptic and strictly increasing in r satisfying some conditions (details).

If $\overline{w}, \underline{w}$ are continuous viscosity super- and subsolutions of (BVP), then $\underline{w} \leq \overline{w}$ on $\overline{D}$.

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Intuition

If $\overline{w}, \underline{w}$ are twice diff, then at $x_0 \in \arg\max_{x \in D}(\underline{w} - \overline{w})(x)$, $\mathrm{d}\underline{w}(x_0) = \mathrm{d}\overline{w}(x_0)$ and $\mathrm{d}^2\underline{w}(x_0) \leq \mathrm{d}^2\overline{w}(x_0)$.

Then

$$\begin{aligned} F(x_0, \underline{w}(x_0), \mathrm{d}\underline{w}(x_0), \mathrm{d}^2\underline{w}(x_0)) \leq 0 &\leq F(x_0, \overline{w}(x_0), \mathrm{d}\overline{w}(x_0), \mathrm{d}^2\overline{w}(x_0)) \\ &= F(x_0, \overline{w}(x_0), \mathrm{d}\underline{w}(x_0), \mathrm{d}^2\overline{w}(x_0)) \\ &\leq F(x_0, \overline{w}(x_0), \mathrm{d}\underline{w}(x_0), \mathrm{d}^2\underline{w}(x_0)) \\ \implies \overline{w}(x_0) \geq \underline{w}(x_0) \qquad &\because F \text{ is strictly increasing in } r. \end{aligned}$$

Hence, if $\underline{w} \leq \overline{w}$ on ∂D, then $\underline{w} \leq \overline{w}$ on $\overline{D}$.

Usefulness: doing comparative statics using HJB!

References for Viscosity Solutions

Crandall, Ishii, & Lions (1992), 'User's guide to viscosity solutions of second order partial differential equations' (doi: 10.1090/S0273-0979-1992-00266-5).

The reference for viscosity solutions.

Definition, ∃, and ! results for a wide variety of cases.

References for Viscosity Solutions

Lions (1983), 'Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations'; Part 1: The dynamic programming principle and applications (doi: 10.1080/03605308308820297); Part 2: Viscosity solutions and uniqueness (doi: 10.1080/03605308308820301) (Part 2 is the most useful).

Flow payoff $f(x, a)$ cont in a, unif cont in x, and bounded; scrap value cont in x.

Ito diffusion $\mathrm{d}X_t = \boldsymbol{\mu}(X_t, a)\,\mathrm{d}t + \boldsymbol{\sigma}(X_t, a)\,\mathrm{d}B_t$, where $\boldsymbol{\mu}, \boldsymbol{\sigma}$ are bounded in x uniformly in a, cont in a.

Main results: V is viscosity soln; $\exists$ and ! of viscosity soln.

Extensions: on domain, relaxing conditions on $\boldsymbol{\mu}, \boldsymbol{\sigma}$; to time-dependent $\boldsymbol{\mu}, \boldsymbol{\sigma}$; and to optimal stopping + control problems.

Caution:

(a) the text is difficult to parse; results given in very general and terse formulations;

(b) it's the 80s and it's math – some results are stated with just sketched proofs (if at all); but these people do know what they're talking about.

References for Viscosity Solutions

Oksendal & Reikvam (1988), 'Viscosity Solutions of Optimal Stopping Problems' (doi: 10.1080/17442509808834137).

Optimal stopping without optimal control, but easy derivation.

Main results: V is viscosity soln; $\exists$ and ! of viscosity soln.

Examples for which smooth pasting doesn't work.

Lions (1986), 'Optimal Control of Reflected Diffusion Processes: An example of state constraints' (doi: 10.1007/BFb0041170)

Ito diffusion with reflection; optimal control problem with discounting.

Main results: V is viscosity soln; $\exists$ and ! of viscosity soln; regularity properties.

Fleming & Soner (2006), *Controlled Markov Processes and Viscosity Solutions*

Ch. 5: *textbook* treatment of viscosity solutions in relation to optimal control.

Pham 09, *Continuous-time Stochastic Control and Optimization with Financial Applications*

Ch. 4: *textbook* treatment of viscosity solutions in relation to optimal control.

Great general purpose reference.

References for Viscosity Solutions

Viscosity in economics:

Faingold & Sannikov (2011 Ecta): reputation in continuous time.

Kaplan, Moll, & Violante (2018 AER): HANK and monetary policy.

Kuvalekar & Lipnowski (2020 AEJMicro): job security and career concerns.

Achdou, Han, Lasry, Lions, & Moll (2021 REStud): income and wealth distribution.

Escudé & Sinander (2023 TE): info design.

Barilla & Gonçalves (2023 TE): creating instability.

References for Viscosity Solutions

Teachings from/for applications:

Math results almost never apply exactly, e.g.

- Flow/terminal payoffs may be discontinuous.
- Optimal control may be discontinuous.
- μ, σ may not be appropriately Lipschitz (e.g. DM can stop process).

In all the papers above, proofs for V being viscosity soln, and existence and uniqueness of viscosity soln mirror one of the references provided and do adjustments whenever necessary.

Often need to relax some particular condition from math refs,
but problem being analysed always has additional structure that can be exploited to make the proofs go through.

A roadmap for adjusting existing results

1. Check how HJB and boundary conditions 'should' look like.
2. Prove version of DPP adapting Lions/Pham/Fleming & Soner.
3. Prove version of comparison principle adapting CIL/Fleming & Soner.
4. Get existence of viscosity solution via Perron's method and uniqueness from comparison principle (again adapting CIL/Fleming & Soner).

   Produce $\overline{w}, \underline{w}$ viscosity super-/subsolutions such that $\overline{w} \geq \underline{w}$ on ∂D.

   Define $w^*(x) := \sup\{w(x) \mid \overline{w} \geq w \geq \underline{w}$ and w is viscosity subsoln of (BVP)$\}$.

   (Given right set of assumptions) w^* is unique viscosity solution.

Overview

1. Conflict and Volatility
   - Choosing Instability
   - A Detour Through Viscosity Solutions
   - Back to Choosing Instability
     - Viscosity Characterisation
     - Properties of Best Responses
     - The Inactive Benchmark
     - Symmetry Properties
   - Equilibria
   - Discussion

Back to Choosing Instability

Notation: $y^+ := \max\{y, \mathbf{0}\}$; $n(x)$: outer normal vector ($n(\mathbf{0}) = -\mathbf{1}, n(\mathbf{1}) = \mathbf{1}$).

Back to Choosing Instability

Notation: $y^+ := \max\{y, 0\}$; $n(x)$: outer normal vector ($n(0) = -1, n(1) = 1$).

Theorem

Value function v_a is unique viscosity solution to Reflected Problem (RP):

$$rv_a(x) - rx - b(x)v_a''(x) - \frac{1}{2rc}\left[v_a''(x)^+\right]^2 = 0 \quad \text{on } (0,1) \qquad \text{(HJB)}$$

$$n(x)v_a'(x) = 0 \quad \text{on } \{0,1\} \qquad \text{(BC)}$$

v_a is continuous and, whenever v_a'' exists, optimal control is:

$$a(x) = \frac{1}{rc}v_a''(x)^+.$$

Viscosity solutions: (i) process can degenerate, fail (BC)
(ii) best responses not in general Lipschitz continuous
(iii) value function not generically differentiable

Rewrite solution to (RP) $v \in \mathcal{C}^2$: $F_a(x, v, v'') = 0$ on $(0,1)$ and $B_a(x, v') = 0$ on $\{0,1\}$

$$F_a(x, v, M) := rv - rx - b(x)M - \frac{1}{2rc}\left[M^+\right]^2 \qquad B_a(x, p) := n(x)p$$

Viscosity Approach

Viscosity **sub**solution to (RP) $\underline{w}$ usc on $[0,1]$:

$\forall \varphi \in \mathcal{C}^2$: x_0 local **max**imiser of $\underline{w} - \varphi$,

$F_a(x_0, \underline{w}(x_0), \varphi''(x_0)) \leq 0$ for $x_0 \in (0,1)$.

$\min\{B_a(x_0, \varphi'(x_0)), F_a(x_0, \underline{w}(x_0), \varphi''(x_0))\} \leq 0$ for $x_0 \in \{0,1\}$.

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Viscosity solution := viscosity sub- and supersolution.

Viscosity Characterisation

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v_a is continuous and, whenever v_a'' exists, optimal control is $a(x) = \frac{1}{rc}v_a''(x)^+$.

Proof sketch

(i) Making sure these are 'right' BC for reflected problem $\rightarrow$ Lions (1986).

(ii) $\sigma(a, X_t) = \sqrt{2(a + b(X_t))}$ non-Lipschitz in $a \in \mathbb{R}_+$ and possibly not in X_t neither! Existing results won't apply.

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(v) $\exists$ and ! via Perron's method + comparison principle: pointwise sup v-subsolns is v-soln

Characterising Best Responses: Satisficing Threshold

Theorem

There are two thresholds $0 < \underline{x}_a \leq \overline{x}_a \leq 1$ such that

(i) on $[0, \underline{x}_a)$ (beneficial instability region) $v_a(x) > x$ and $a(x) \propto v_a''(x) > 0$;

.

Statements about $v_a''(x)$ apply only at points of differentiability.

Characterising Best Responses: Satisficing Threshold

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Proof intuition

Alvarez, Lasry, & Lions (1997, Lemma 1): if for $u \in USC(\overline{D})$ and $\forall x \in D$ and $\forall (p, M) \in J_D^+ u(x)$, $M \geq 0$, then u is convex in D.

Adjust to prove convex-concave structure from (HJB): $v_a(x) \geq (\leq)\, x$ iff $v_a'' \geq (\leq)\, 0$

$$rv_a(x) - rx - b(x)v_a''(x) - \frac{1}{2rc}\left[v_a''(x)^+\right]^2 = 0 \quad \text{on } (0,1). \tag{HJB}$$

Rule out convex kinks via viscosity supersoln requirements; $\mathcal{C}^2$ via continuity of $\overline{J}^{2,\pm}$.

Characterising Best Responses: Satiscifing Threshold

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Value function v_a is increasing and $\mathcal{C}^2$ everywhere, except possibly at $\overline{x}_a$

- Beneficial instability region always nonempty:

  Always push back from extremely unfavourable status quo.

  But can be arbitrarily small.

- Lower threshold $\underline{x}_a$: captures ability to profit from instability.

  Beyond $\underline{x}_a$, too costly to generate instability (option value):

    opponent's strategy may prevent achieving durable gains.

Inactive Benchmark

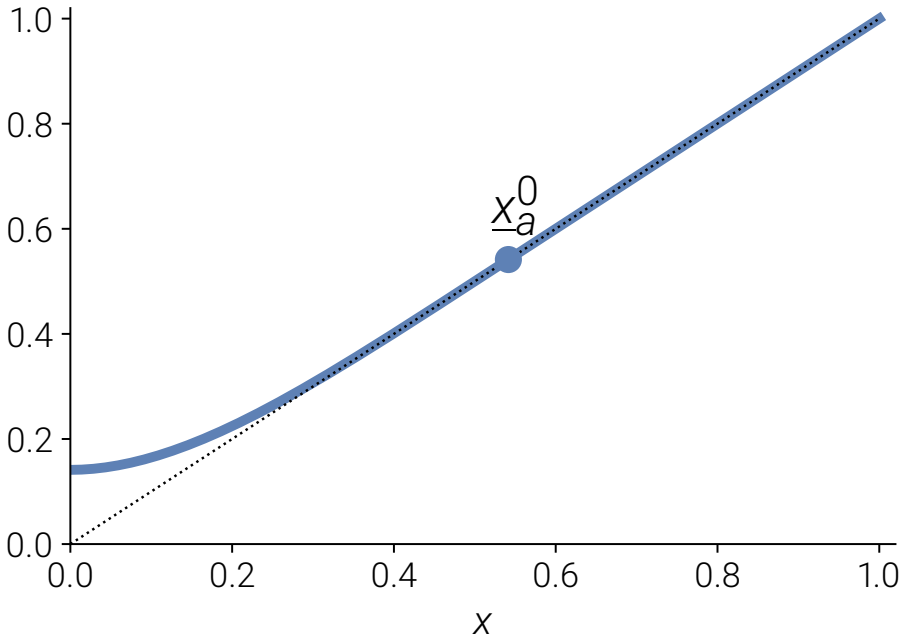
Inactive Benchmark: other player is inactive ($b \equiv 0$).

Will be "right" benchmark for equilibrium.

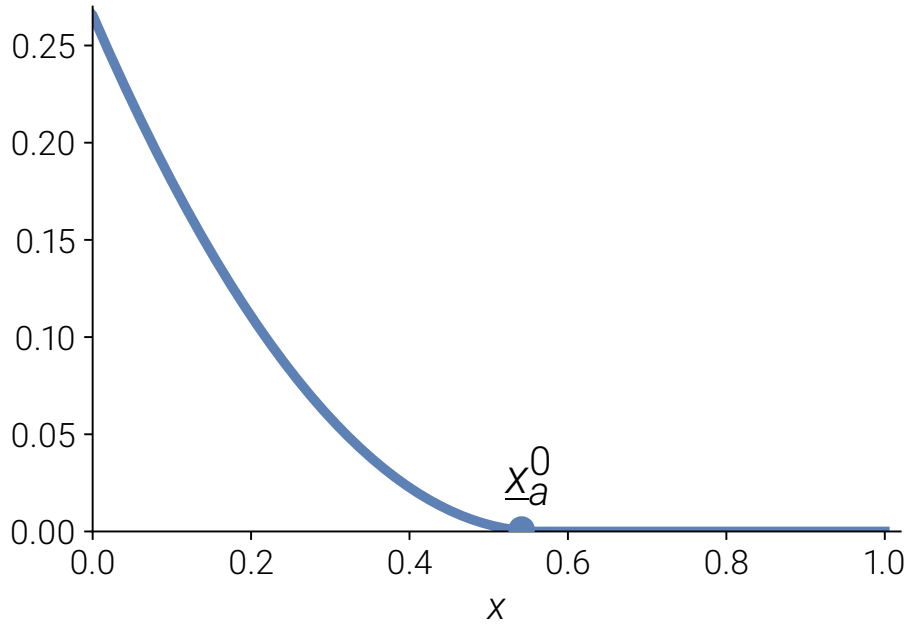
HJB has nice interpretation:

$$\underbrace{r(v_a(x) - x)}_{\text{improvement on the status quo}} = \underbrace{\frac{1}{2rc}[v_a''(x)_+]^2}_{\text{option value}} = \underbrace{\frac{rc}{2}a(x)^2}_{\text{instantaneous cost of control}}$$

Inactive Benchmark



(a) Value Function v_a^0

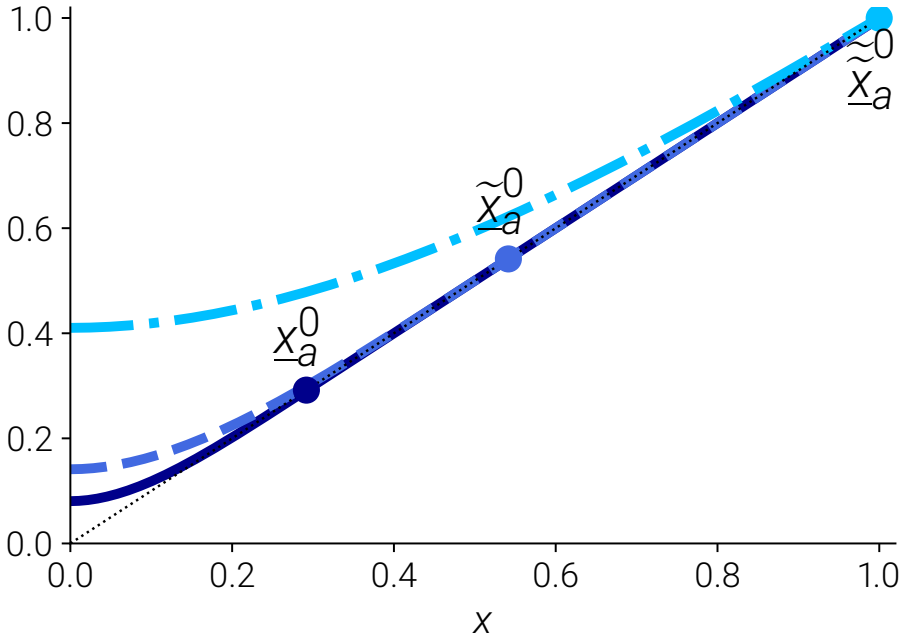


(b) Optimal Strategy a^0

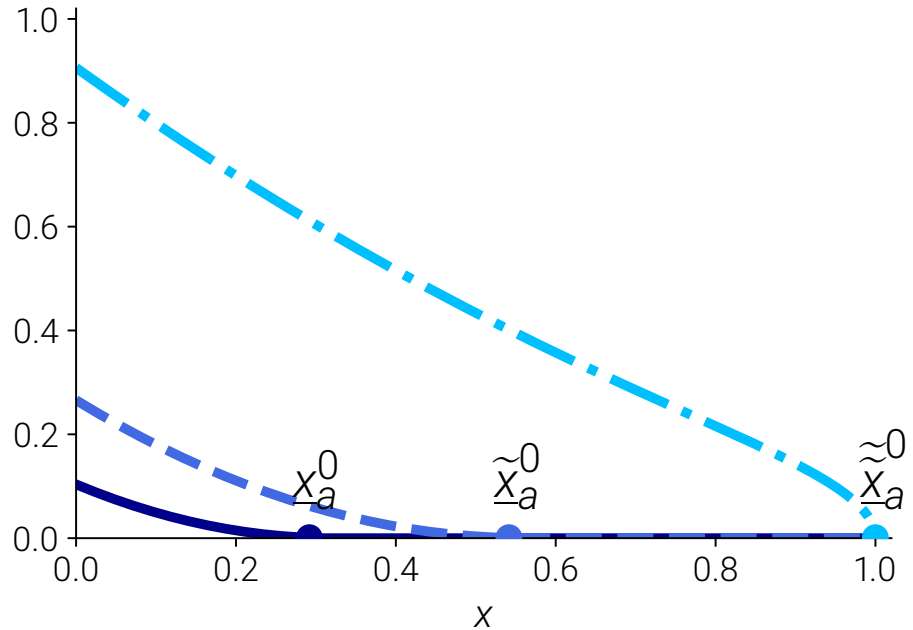
$v_a^0, a^0, \underline{x}_a^0$: value function, optimal control, threshold.

$b(1) = 0$: no detrimental instability region (iff condition).

Inactive Benchmark: Comparative Statics



(a) Value Function v_a^0



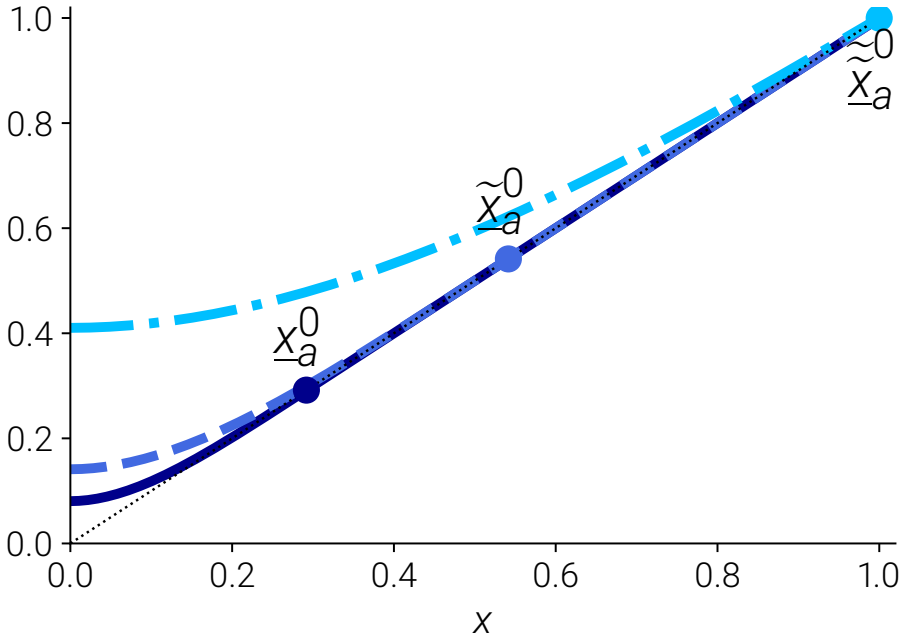
(b) Optimal Strategy a^0

$\uparrow r$: lower mg benefit from option value; $\uparrow c$: higher mg cost to instability.

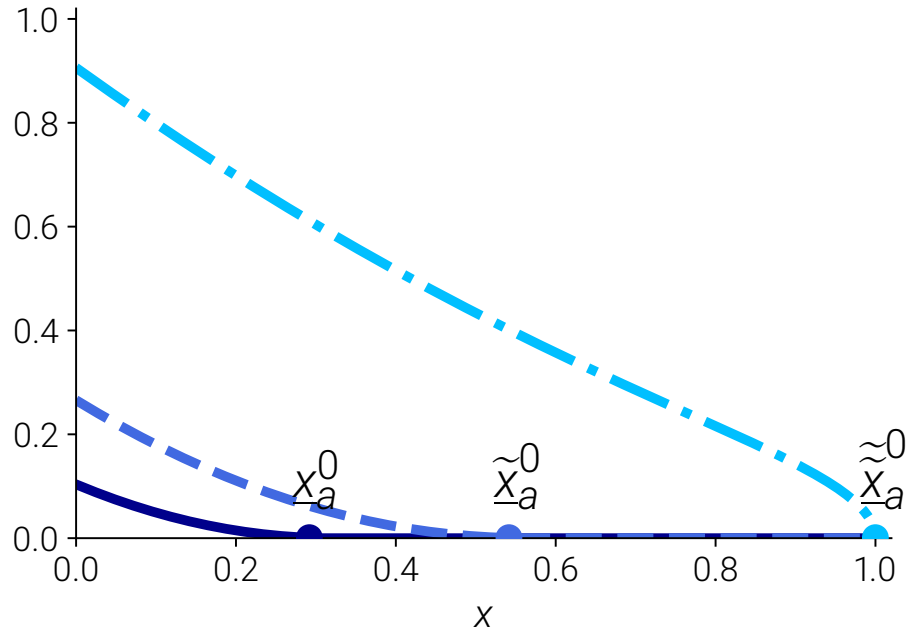
higher threshold ($\underline{x}_a^0$); value and instability increase pointwise.

Can achieve $\underline{x}_a^0 = 1$.

Inactive Benchmark: Comparative Statics



(a) Value Function v_a^0



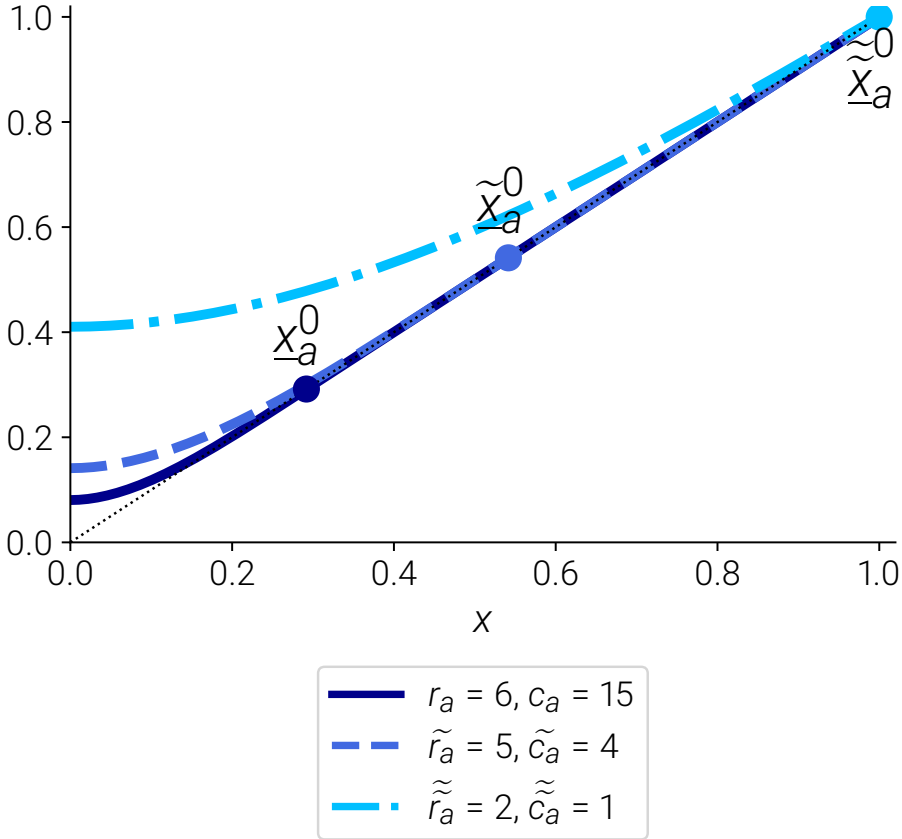
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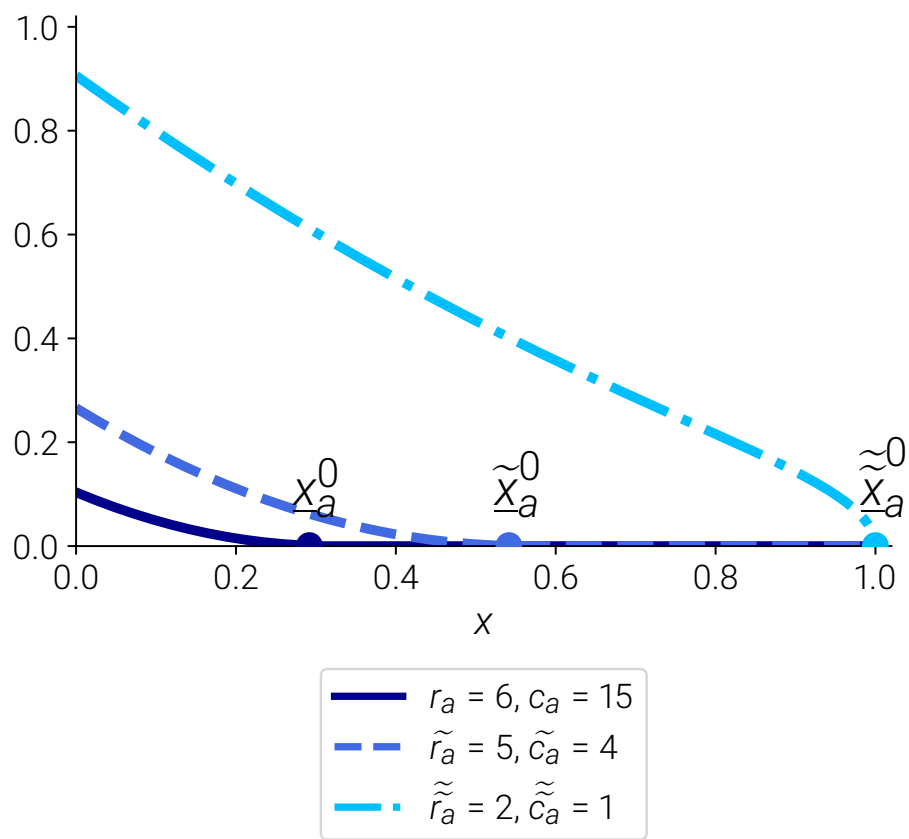
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Can achieve $\underline{x}_a^0 = 1$.

Inactive Benchmark: Comparative Statics



(a) Value Function v_a^0



(b) Optimal Strategy a^0

Intuition: $0 = F_a(x, v_a, p, M) = r(v_a - x) - \frac{1}{2rc}(v_a''(x)^+)^2$.

If $\tilde{r}, \tilde{c} > r, c$, then v_a becomes supersolution $\implies$ v_a pointwise higher than $\tilde{v}_a$ (comparison principle).

Inactive Benchmark: Actively Reflecting Upper Bound

Proposition

In inactive benchmark, optimal control a:

(i) is convex if $\underline{x}_a^0 < 1$;

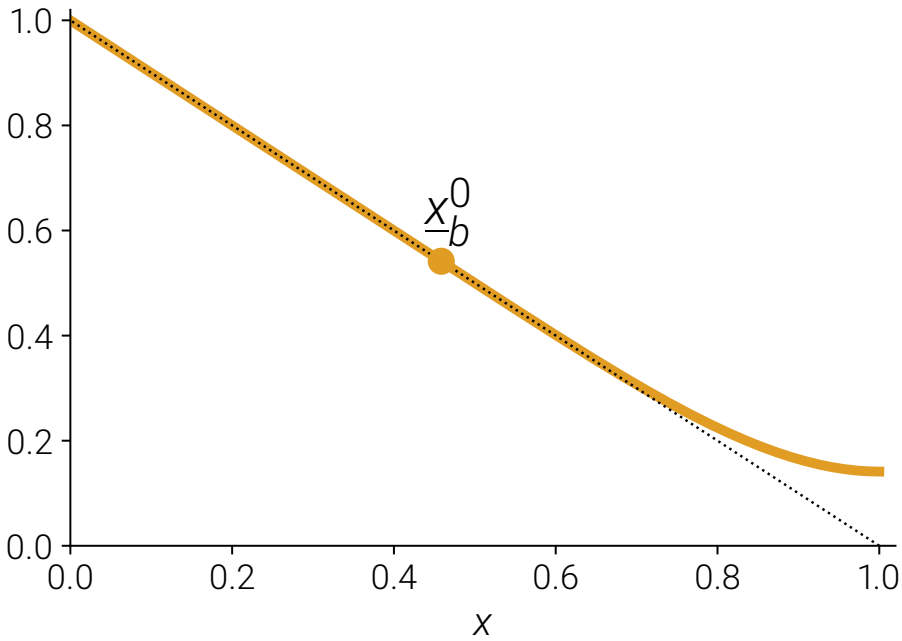
(ii) is convex-concave only if $\underline{x}_a^0 = \bar{x}_a^0 = 1$, in which case $a^{0\prime}(x) \to -\infty$ as $x \to 1$.

Non-actively reflecting upper bound $\implies$ control convex throughout.

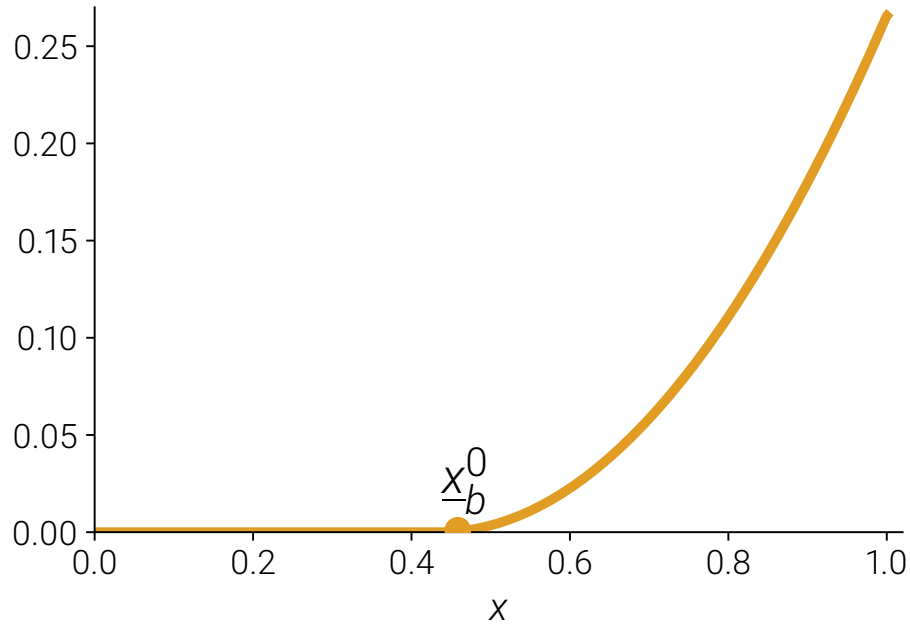
Actively reflecting upper bound $\implies$ control concave near upper bound.

Instability vanishes quickly.

Symmetry Properties



(a) Value Function v_b^0



(b) Optimal Strategy b^0

Case for player B symmetric: swap x for $\mathbf{1} - x$.

Overview

1. Conflict and Volatility
   - Choosing Instability
   - A Detour Through Viscosity Solutions
   - Back to Choosing Instability
   - Equilibria
     - Properties of Equilibria
     - Accommodating Equilibria
     - Deterrence Equilibria
   - Discussion

Decoupling of Equilibrium Strategies

Proposition

At any equilibrium, there are $0 < \underline{x} \leq \overline{x} < 1$:

(i) on $[0, \underline{x})$ ((*A*'s *instability region*)) $a(x) > 0 = b(x)$;

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Moreover, a is strictly decreasing on $[\mathbf{0}, \underline{x})$ and b is strictly increasing on $(\overline{x}, \mathbf{1}]$.

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Proof intuition

(i) Best responses are such that $a(0), b(1) > 0$.

(ii) States at which player generates instability connected.

(iii) Players cannot both expect to benefit from instability: instability sets disjoint.

(iv) Monotone strategies induce monotone best responses.

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Moreover, a is strictly decreasing on $[0, \underline{x})$ and b is strictly increasing on $(\overline{x}, 1]$.

Instability if status quo deemed too unfavourable by either player.

Stability region ($a(x) = b(x) = 0$) corresponds to more moderate states.

Players more conservative (create less instability) the more favourable is status quo.

Constructing Equilibria

Previous results do not guarantee existence

... But we can construct equilibrium directly!

Two classes of equilibria depending on parameters.

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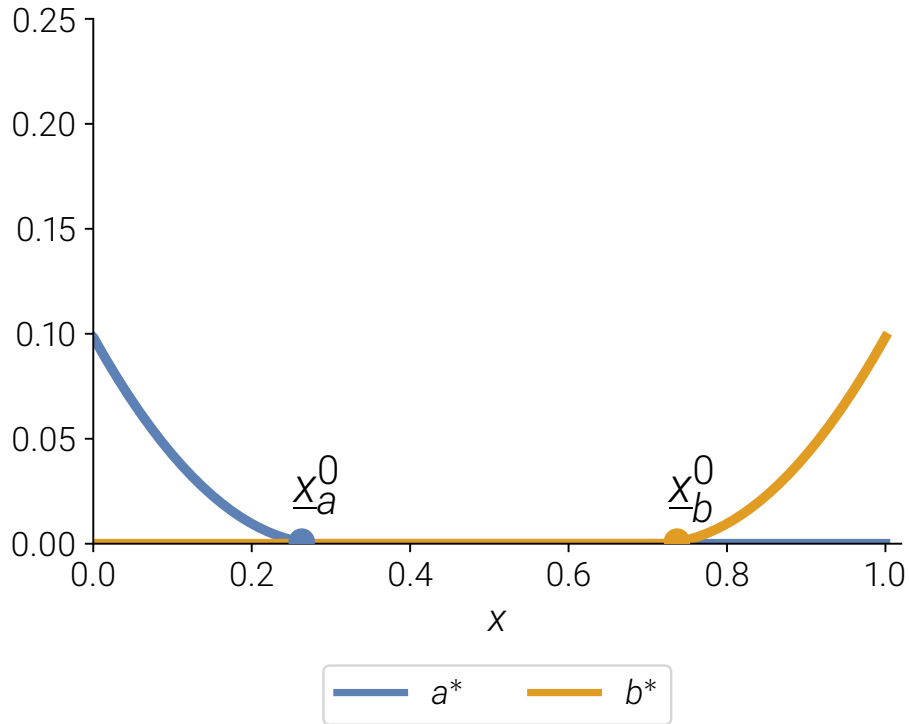
If benefits from instability are low enough ($\underline{x}_a^0 \leq \underline{x}_b^0$), players can (partially) attain the inactive benchmark's payoff.

$\Longrightarrow$ **Accommodating Equilibrium:**

Eqm behaviour is accommodating towards opponent's aspirations to obtain a better outcome by creating some instability.

Players never 'push back' against one another.

Accommodating Equilibria



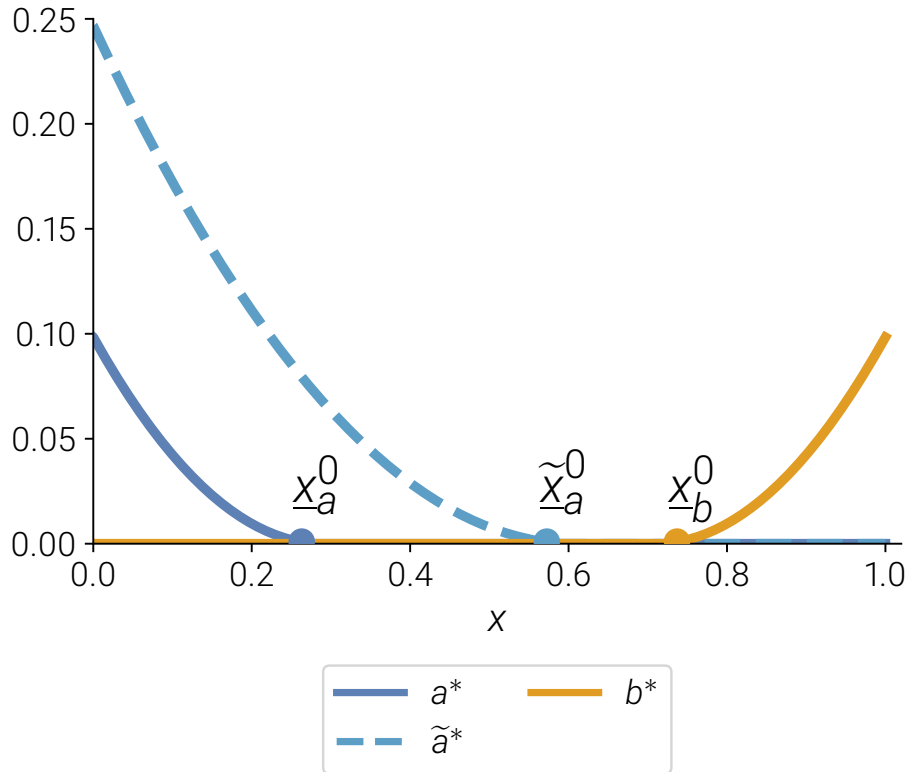
Theorem

If $\underline{x}_a^0 \leq \underline{x}_b^0$, there is unique equilibrium: $(a^*, b^*) = (a^0, b^0)$.

Behave as in inactive benchmark; at $[\underline{x}_a^0, \underline{x}_b^0]$ status quo prevails.

Unique eqm if players too impatient/have high enough costs to instability.

Accommodating Equilibria



Comparative statics on impatience/costs carries over to eqm.

Lower cost/impatience ($\downarrow c_a, r_a$) $\implies$ Greater instability ($\uparrow \underline{x}_a^0$).

Shrinks stable region.

Crossing Thresholds

What if $\underline{x}_a^0 > \underline{x}_b^0$?

Region where both players want to create instability if opponent were passive.

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$$a'(x) \to -\infty \quad \text{as} \quad x \to \underline{x}.$$

Observations imply players use instability to actively 'push back' opponent.

Deters opponent from wanting to destabilise status quo.

$\Longrightarrow$ **Deterrence Equilibrium**.

Constructive Characterisation

What if $\underline{x}_a^0 > \underline{x}_b^0$?

Suggests constructive method:

Pick $\overline{x} \in [\underline{x}_b^0, \underline{x}_a^0]$.

Solve for inactive benchmark as if there is reflection on $\overline{x}$.

Solve for value function in region where player is inactive.

Piece together functions and check whether it is a soln to the original problem.

Does this method work?

Are there other kinds of equilibria?

Deterrence Equilibria

Theorem

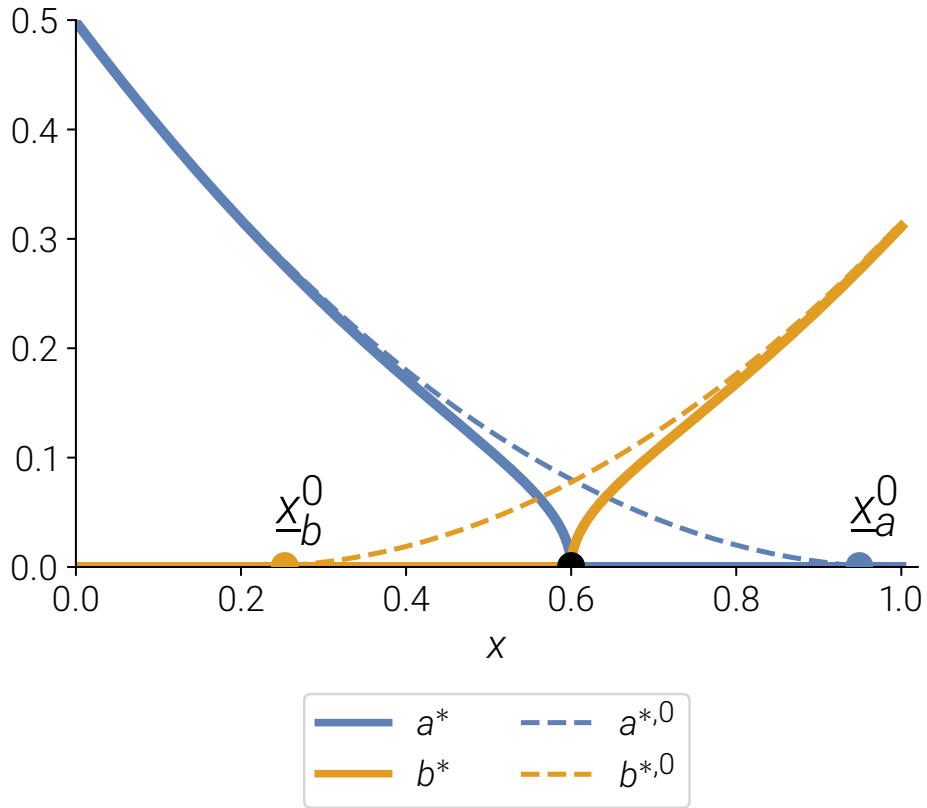
If $\underline{x}_a^0 > \underline{x}_b^0$,

(i) Every equilibrium is characterised by a unique threshold $\underline{x} = \overline{x} \in [\underline{x}_b^0, \underline{x}_a^0] \cap (0, 1)$ corresponding to its single stable status quo.

(ii) Every $\overline{x} \in [\underline{x}_b^0, \underline{x}_a^0] \cap (0, 1)$ corresponds to an equilibrium.

(iii) Equilibrium strategies obtained by solving inactive benchmark on restricted domains $[0, \overline{x}]$ and $[\overline{x}, 1]$.

Deterrence Equilibria

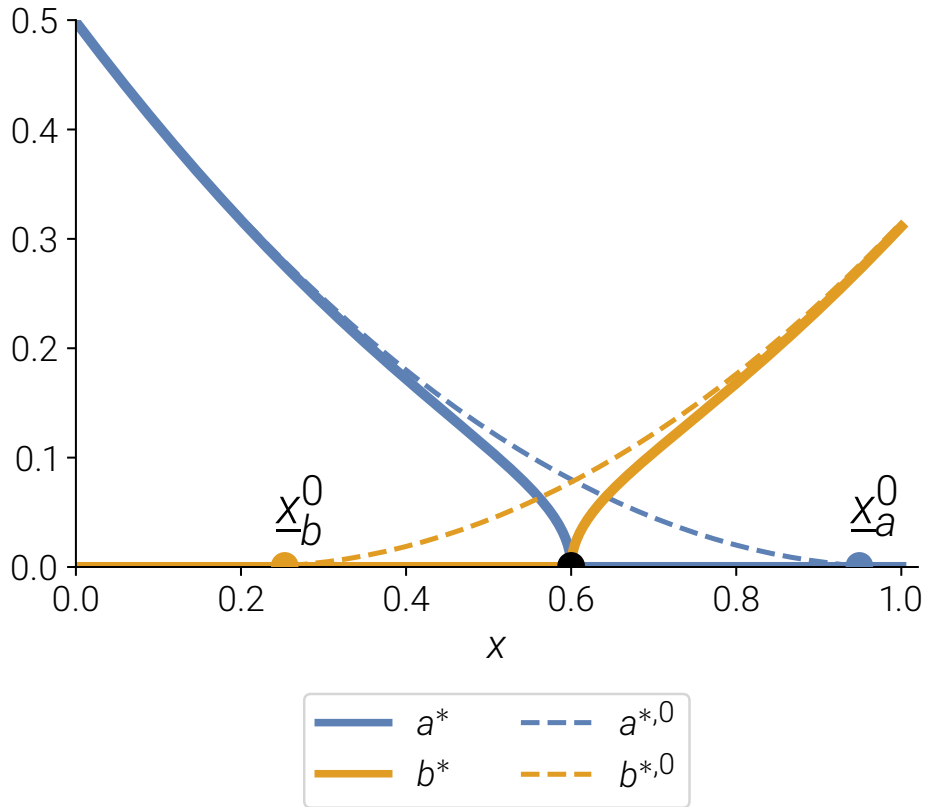


Deterrence: Players actively “push back”.

As if eqm stable state actively reflecting: convex-concave controls.

‘Concave kink’ in value function.

Deterrence Equilibria



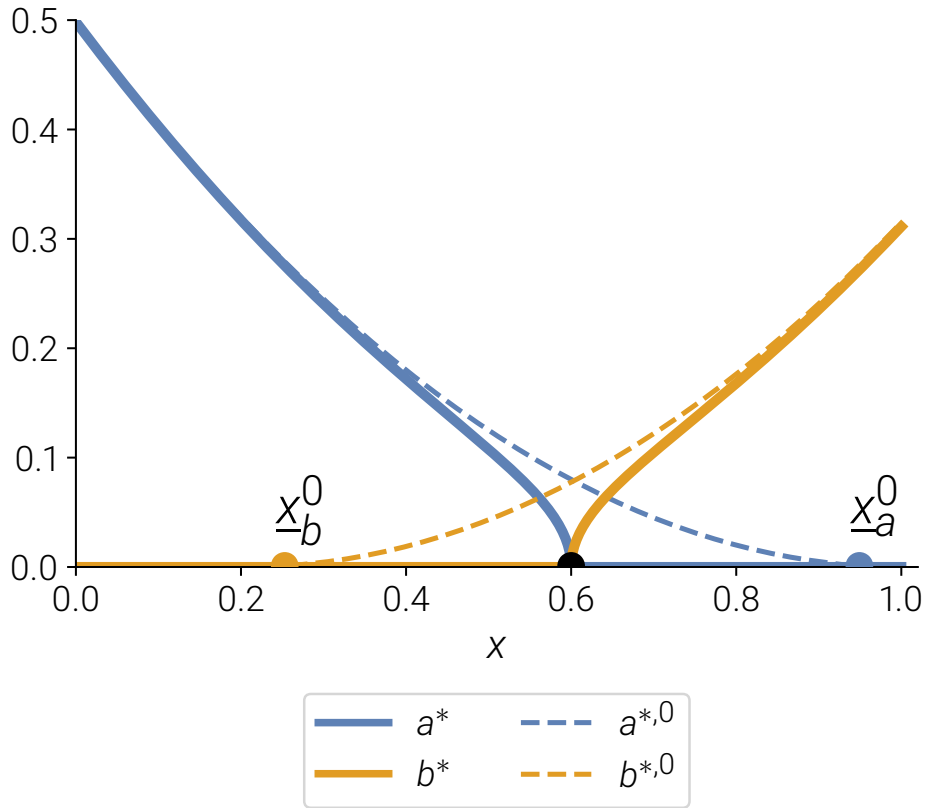
Inactive benchmark thresholds $\underline{x}_a^0, \underline{x}_b^0$ capture players' threat power.

At any *eqm*, $v_a \leq v_a^0$ and $v_b \leq v_b^0$ (not true for arbitrary opponent's strategies).

Implies (i) generate less instability in eqm than at inactive benchmark ($a^* \leq a^0$)

(ii) satisficing target less ambitious ($\underline{x} \leq \underline{x}_a^0, \quad \underline{x}_b^0 \leq \overline{x}$).

Deterrence Equilibria



Inactive benchmark thresholds $\underline{x}_a^0, \underline{x}_b^0$ capture players' threat power.

Multiple eqa, each pinned-down by $\overline{x} \in [\underline{x}_b^0, \underline{x}_a^0] \cap (0, 1)$.

Extreme states can never be stable: underdog will always push back.

Reduces comparative statics on set of eqa to comparative statics on $\underline{x}_a^0, \underline{x}_b^0$.

Discussion

Exogenous instability:

Exact same characterisation of best responses.

Existence of eqm via fixed-point argument (Schauder FPThm).

Unique state $\overline{x} = \underline{x}$ such that $a^*(\overline{x}) = b^*(\overline{x}) = \mathbf{0}$.

Players perpetually creating instability (mechanically).

Costs to instability: results can be extended to general smooth, strictly convex costs.

Absorbing boundaries or unbounded payoffs:

No incentive to generate instability: risk-neutral players, no option value.

Terminal payoff instead of flow payoff:

Create more instability the more favourable the state;

reminiscent of Moscarini & Smith (2001 Ecta).

Conflict and Volatility

Duarte Gonçalves

University College London

Topics in Economic Theory

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Gonçalves, Speed, Accuracy, and Complexity.

Theorem (7.9 CIL 92)

Let $D \subset \mathbb{R}^n$ be open, convex, and bounded and F be proper degenerate elliptic satisfying some conditions (details).

If $\overline{w}, \underline{w}$ are continuous viscosity super- and subsolutions of (BVP), then $\underline{w} \leq \overline{w}$ on $\overline{D}$.

Conditions in the theorem:

(7.12) $\overline{D}$ convex, compact subset of $\mathbb{R}$ with nonempty interior; require $\mathcal{C}^1$ ∂D for $\mathbb{R}^n$.

(7.14) $\exists \gamma > 0$ s.t. $\gamma(r-s) \leq F(x,r,p,M) - F(x,s,p,M)$ for $r \geq s$ and $(x,p,M) \in \overline{D} \times \mathbb{R}^n \times \mathcal{S}^n$.

(7.15) $\exists \varepsilon > 0$ such that $|F(x,r,p,M) - F(x,r,q,N)| \leq \psi(|p-q| + ||M-N||)$ for all $p, q \in \mathbb{R}^n$, $M, N \in \mathcal{S}^n$, and all $x \in \overline{D}$ sat. $||x - x_0|| < \varepsilon$ for $x_0 \in \partial D$, given some $\psi : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{\infty\}$ satisfying $\psi(0^+) = 0$.

(7.16) $\exists \psi : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{\infty\}$ satisfying $\psi(0^+) = 0$ s.t.
$F(y,r,p,N) - F(x,r,p,M) \leq \psi(\alpha|x-y|^2 + |x-y|(|p|+1))$ for $x, y \in \overline{D}$, $r \in \mathbb{R}$, $p \in \mathbb{R}^n$, and all $\alpha > 0$, $M, N \in \mathcal{S}^n$ s.t.

$$-3\alpha \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} M & 0 \\ 0 & -N \end{pmatrix} \leq 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$$