

Entry, Exit, and Strategic Entry

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Topics in Economic Theory

Buying a Product

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Read another review? Try it in a shop? Ask someone who has one?

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When is it worth waiting?

Same question for investment, entry, exit, and filing a patent.

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Time is often observed, data that comes for free. What can we make of it?

Are there patterns that informative about market forces or private information?

Need theory to make sense of the data.

Overview

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2. Entry and Exit: Hysteresis
3. Strategic Stopping: Races and Contests

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An Irreversible Investment

McDonald & Siegel (1986 QJE); Dixit & Pindyck (1994, ch. 5)

Pay $I > 0$ once to acquire a project worth X_t .

Project value follows geometric Brownian motion:

$$dX_t = \mu X_t dt + \sigma X_t dB_t, \quad \sigma > 0, \quad r > \max\{\mu, 0\}.$$

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May abandon for zero. Scrap value $v(x) = (x - I)^+$ and

$$V(x) = \sup_{\tau} \mathbb{E}_x[e^{-r\tau} v(X_{\tau})].$$

Allow $\tau = \infty$ with zero payoff. This is equivalent to the investment problem with payoff $X_{\tau} - I$ and the option never to invest.

Positive net present value does not settle whether to invest *now*.

Project Value and the Cost of Delay

Itô's formula gives

$$X_t = xZ_t, \quad Z_t = \exp \left((\mu - \sigma^2/2)t + \sigma B_t \right).$$

Thus $X_t > 0$ and $\mathbb{E}_x X_t = xe^{\mu t}$.

The generator is $\mathcal{L}\phi = \mu x\phi' + \sigma^2 x^2 \phi''/2$.

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The generator is $\mathcal{L}\phi = \mu x\phi' + \sigma^2 x^2 \phi''/2$.

Since $r > \mu$, $e^{-rt}X_t$ is a nonnegative supermartingale.

Waiting sacrifices expected discounted project value but postpones the sunk cost and preserves the option to avoid bad outcomes.

The problem is stationary and Markovian: the decision depends on current x .

Why the Stopping Region Must Be One-Sided

At $x \leq l$, immediate investment loses money or breaks even.

For every $x > 0$, waiting a fixed positive time gives a positive probability of $X_t > l$, hence $V(x) > 0$.

Therefore no stopping at $0 < x \leq l$: even abandonment destroys a valuable option.

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Therefore no stopping at $0 < x \leq l$: even abandonment destroys a valuable option.

Couple initial states using the same Z , so $X_t^x = xZ_t$.

For each Brownian stopping time τ , the expected payoff is increasing and convex in x .

Moreover, for $y > x$,

$$0 \leq \mathbb{E}[e^{-r\tau}(yZ_\tau - l)^+] - \mathbb{E}[e^{-r\tau}(xZ_\tau - l)^+] \leq y - x,$$

$$\text{because } \mathbb{E}[e^{-r\tau}Z_\tau \mathbf{1}_{\{\tau < \infty\}}] \leq 1.$$

A Single Kink Gives an Upper Stopping Region

Proposition

V is increasing, convex, and 1-Lipschitz. On $[l, \infty)$, $V(x) - (x - l)$ is nonnegative, convex, and nonincreasing. Thus if stopping is optimal at x , it is optimal at every $y > x$.

Proof

Taking the supremum preserves convexity, monotonicity, and the common Lipschitz bound. For $y > x \geq l$,

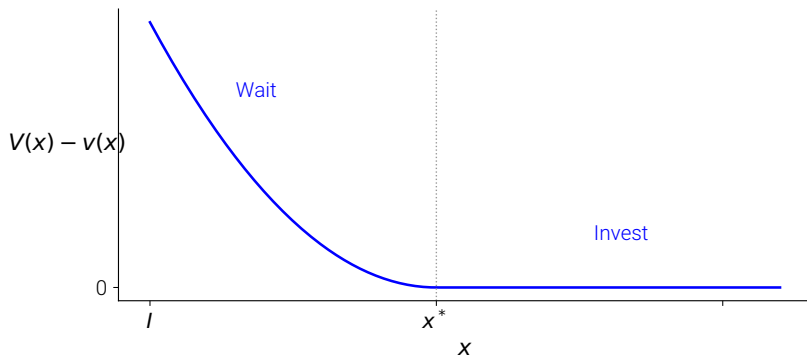
$$V(y) - (y - l) \leq V(x) - (x - l).$$

Once the nonnegative option premium reaches zero, it stays zero.

The one-kink scrap value and the process structure deliver an upper threshold; its shape is not a guess.

This argument allows an empty stopping region. The solution below establishes contact at a finite x^* .

The Option Premium Vanishes Only Once



Above I , another pound raises scrap value by exactly one pound.

It raises the value of the investment option by at most one pound.

The gain from retaining flexibility therefore falls until exercise becomes optimal.

Locating the Investment Threshold

Below an upper exercise boundary, the candidate must satisfy

$$\frac{1}{2}\sigma^2 x^2 \tilde{V}'' + \mu x \tilde{V}' - r \tilde{V} = 0.$$

Power solutions x^β have

$$\frac{1}{2}\sigma^2 \beta(\beta - 1) + \mu\beta - r = 0, \quad \beta_- < 0 < 1 < \beta_+.$$

Since $0 \leq V(x) \leq x$, the negative-power term is excluded near zero.

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Since $0 \leq V(x) \leq x$, the negative-power term is excluded near zero.

Write $\beta = \beta_+$. Value matching and smooth fit give

$$A(x^*)^\beta = x^* - I, \quad A\beta(x^*)^{\beta-1} = 1.$$

$$\text{Hence } x^* = \frac{\beta}{\beta-1}I > I.$$

Verifying the Investment Decision

Define

$$\tilde{V}(x) = \begin{cases} (x^* - I)(x/x^*)^\beta, & 0 < x < x^*, \\ x - I, & x \geq x^*. \end{cases}$$

It is C^1 , nonnegative, and dominates v ; the convex power curve lies above its tangent $x - I$.

Verifying the Investment Decision

Define

$$\tilde{V}(x) = \begin{cases} (x^* - l)(x/x^*)^\beta, & 0 < x < x^*, \\ x - l, & x \geq x^*. \end{cases}$$

It is C^1 , nonnegative, and dominates v ; the convex power curve lies above its tangent $x - l$.

Below x^* , $(\mathcal{L} - r)\tilde{V} = 0$.

Above it,

$$(\mathcal{L} - r)(x - l) = rl - (r - \mu)x \leq 0,$$

$$\text{since } (r - \mu)x^* - rl = \frac{1}{2}\sigma^2\beta l > 0.$$

The nonnegative discounted candidate is a supermartingale, so no feasible rule pays more than $\tilde{V}$.

The Threshold Rule Attains the Bound

Let $\tau^* = \inf\{t : X_t \geq x^*\}$, with payoff zero if never hit.

For $x < x^*$, the discounted hitting-time transform is

$$\mathbb{E}_x[e^{-r\tau^*} \mathbf{1}_{\{\tau^* < \infty\}}] = \left(\frac{x}{x^*}\right)^\beta.$$

To see this, stop the local martingale $e^{-rt}X_t^\beta$ at $t \wedge \tau^*$; it is bounded by $(x^*)^\beta$.

On paths not yet stopped, $e^{-rt}X_t^\beta \leq e^{-rt}(x^*)^\beta \rightarrow 0$.

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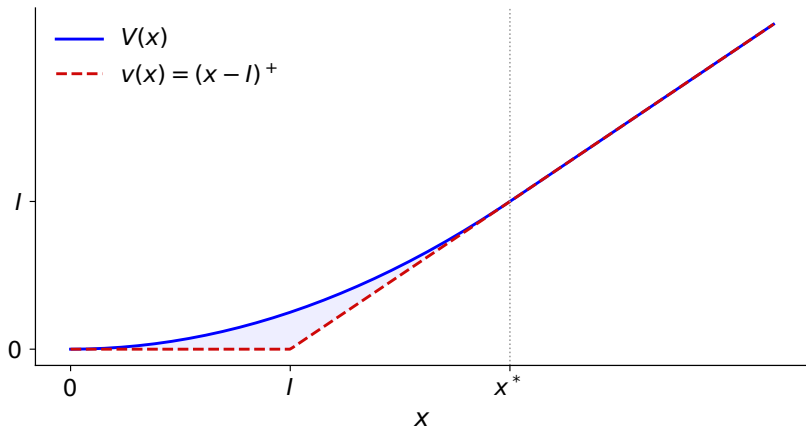
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Thus the threshold rule earns exactly $\tilde{V}(x)$, and $V = \tilde{V}$.

Invest when $x \geq x^*$; otherwise keep the option alive.

Positive NPV Is Not Enough



At $I < x < x^*$, investment is profitable but waiting is better.

Exercise sacrifices $V(x) - v(x)$, the value of responding to future news.

The trigger exceeds cost by $x^* - I = I/(\beta - 1)$.

Greater Uncertainty Raises the Investment Trigger

Let $q(\beta) = \frac{1}{2}\sigma^2\beta(\beta - 1) + \mu\beta - r$.

At its positive root, $q_\beta > 0$, so

$$\frac{\partial \beta}{\partial \sigma^2} = -\frac{\beta(\beta - 1)}{2q_\beta} < 0, \quad \frac{\partial x^*}{\partial \sigma^2} = -\frac{1}{(\beta - 1)^2} \frac{\partial \beta}{\partial \sigma^2} > 0.$$

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More uncertainty increases the value of retaining flexibility under irreversible investment.

The consumer waits for stronger evidence; the investor waits for a higher project value.

A higher trigger need not mean a longer expected delay under every process: volatility also changes the speed of reaching it.

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Entry and Exit Decisions

Setup (Dixit 1989 JPE)

Output price: $dP_t = \mu P_t dt + \sigma P_t dB_t$, $r > \max\{\mu, 0\}$.

Active firm earns flow profit $P_t - w$, $w > 0$.

Idle firm earns 0.

Entry costs $I > 0$; exit costs $E \geq 0$; re-entry is possible.

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Entry costs $I > 0$; exit costs $E \geq 0$; re-entry is possible.

$V_0(p)$: value when idle; $V_1(p)$: value when active.

Switching changes the regime, not the price process.

Entry and exit are linked stopping problems.

Two Linked Stopping Equations

Idle: wait or take scrap value $v_0 = V_1 - I$ by entering.

$$\max\{V_1 - I - V_0, (\mathcal{L} - r)V_0\} = 0.$$

Active: operate or take scrap value $v_1 = V_0 - E$ by exiting.

$$\max\{V_0 - E - V_1, (\mathcal{L} - r)V_1 + p - w\} = 0.$$

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With linear operating profits and sunk switching costs, consider entry at p_H and exit at $p_L < p_H$; verify both regimes below.

For $p_L < p < p_H$, action depends on whether the firm is already active.

The state is (p, regime) , not just p .

Solving for the Entry and Exit Triggers

Let $\beta_- < 0 < 1 < \beta_+$ solve the same characteristic equation as before.

In each regime's continuation region,

$$V_0(p) = Ap^{\beta_+}, \quad V_1(p) = \frac{p}{r - \mu} - \frac{w}{r} + Bp^{\beta_-}.$$

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Value matching and smooth fit give four equations:

$$\begin{aligned} V_0(p_H) &= V_1(p_H) - I, & V'_0(p_H) &= V'_1(p_H), \\ V_1(p_L) &= V_0(p_L) - E, & V'_1(p_L) &= V'_0(p_L). \end{aligned}$$

Solve for A, B, p_L, p_H ; check the regime inequalities.

If exit is too costly, an interior exit trigger need not exist.

Hysteresis

Consider $w > rE$, so exit can be worthwhile at low prices.

At the entry boundary, waiting must be unattractive to the idle firm:

$$(\mathcal{L} - r)(V_1 - l) = -(p - w) + rl \leq 0 \quad \implies \quad p_H \geq w + rl.$$

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At the exit boundary, continued operation must be unattractive:

$$(\mathcal{L} - r)(V_0 - E) + p - w = rE + p - w \leq 0 \implies p_L \leq w - rE.$$

With nondegenerate uncertainty and interior triggers, the standard solution has strict inequalities.

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Hysteresis: the same current price can sustain an active or an idle firm, depending on past entry and exit.

Entry and Exit: A Numerical Example

$$r = .05, \mu = 0, \sigma = .20, w = 1, l = 2, E = 1.$$

Solving the four boundary equations gives

$$p_L \simeq 0.651, \quad p_H \simeq 1.675.$$

Compare with $w - rE = .95$ and $w + rl = 1.10$.

	$p < p_L$	$p_L < p < p_H$	$p > p_H$
Currently idle	Wait	Wait	Enter
Currently active	Exit	Operate	Operate

Sunk costs and uncertainty create a region of inaction.

A temporary price shock can have a persistent effect on activity.

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 - R&D Competition
 - Gambling in Contests

Strategic Delay in R&D

Weeds (2002 REStud)

Two firms choose when to initiate irreversible research.

Patent value follows geometric Brownian motion.

Active research succeeds at a Poisson arrival rate.

The first discovery wins the patent.

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The first discovery wins the patent.

Waiting preserves flexibility; investing can secure a lead.

A rival's entry reduces the value of an active project.

Competition can induce preemption, but also sustain joint delay.

The monopoly stopping threshold alone does not determine equilibrium.

Leader and Follower Values

Compute payoffs backwards:

- Value when both firms conduct research.

- Follower's option to initiate research.

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Where $V_L(x) > V_F(x)$, each firm would prefer to move first.

Whether delaying together is sustainable also requires checking deviations.

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Smooth fit applies at an optimally chosen follower boundary.

At the rival's boundary, the leader's value must match; the leader does not choose that boundary, so do not impose its smooth-fit condition there.

Research Success as a Random Arrival

Symmetric race: entry cost I , research flow cost c_R , success rate λ per active firm, independent of patent value X .

In a short interval h , one active firm's success probability is $\lambda h + o(h)$.

With two active firms, the race ends at total rate 2λ .

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A firm's value after both have entered is

$$\begin{aligned} A_2(X) &= \mathbb{E}_X \int_0^\infty e^{-(r+2\lambda)t} (\lambda X_t - c_R) dt \\ &= \frac{\lambda X}{r + 2\lambda - \mu} - \frac{c_R}{r + 2\lambda}. \end{aligned}$$

Assume $r > \max\{\mu, 0\}$. The rival's possible success discounts future gains.

The Follower's Stopping Problem

The follower can take scrap value $v_F = A_2 - I$. While it waits, leader success ends the opportunity at rate λ .

Therefore, before the follower's entry threshold x_F ,

$$(\mathcal{L} - r - \lambda)V_F = 0.$$

Write $V_F(x) = Kx^\gamma$, where $\gamma > 1$ solves

$$\frac{1}{2}\sigma^2\gamma(\gamma - 1) + \mu\gamma - (r + \lambda) = 0.$$

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Set $V_F(x_F) = A_2(x_F) - I$ and $V'_F(x_F) = A'_2(x_F)$:

$$x_F = \frac{\gamma}{\gamma - 1} \left(I + \frac{C_R}{r + 2\lambda} \right) \frac{r + 2\lambda - \mu}{\lambda}.$$

The stopping method survives; the rival changes both the payoff and the effective discount rate.

The Leader Anticipates Entry

Let $A_1(x)$ be the active leader's value before follower entry.

Until X reaches x_F ,

$$(\mathcal{L} - r - \lambda)A_1 + \lambda x - c_R = 0.$$

A solution satisfying the natural condition near zero is

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Determine C from $A_1(x_F) = A_2(x_F)$.

Net leader value before initiating research is $V_L(x) = A_1(x) - I$.

These values supply the timing game; choosing equilibrium initiation requires the leader–follower incentive comparisons.

Gambling in Contests

Setup (Seel & Strack 2013 JET)

$n \geq 2$ players privately observe independent processes $X_t^i = x_0 + \mu t + \sigma B_t^i$, $x_0 > 0$.

Choose a possibly randomised stopping time τ_i , with $\mathbb{E}\tau_i < \infty$.

Must stop by bankruptcy: $\tau_i \leq \inf\{t : X_t^i = 0\}$.

Highest stopped value wins a unit prize; ties split it.

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Highest stopped value wins a unit prize; ties split it.

Focus on $\mu \leq 0$.

Given atomless rival cdf G , the scrap value is $v(x; G) = G(x)^{n-1}$.

Rank incentives change the stopping payoff.

A Driftless Contest

Let $\mu = 0$. Suppose symmetric equilibrium has upper support $\bar{x}$.

Indifference on the support requires a harmonic winning probability:

$$\frac{1}{2}\sigma^2 v'' = 0, \quad v(0) = 0, \quad v(\bar{x}) = 1.$$

Thus $v(x) = x/\bar{x}$ on $[0, \bar{x}]$.

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Thus $v(x) = x/\bar{x}$ on $[0, \bar{x}]$.

Symmetry: $v(x_0) = 1/n$, hence $\bar{x} = nx_0$.

$$G(x) = \left(\frac{x}{nx_0} \right)^{1/(n-1)}, \quad 0 \leq x \leq nx_0.$$

A candidate distribution is not yet a stopping strategy.

Constructing an Equilibrium: Two Players

Proposition ($n = 2, \mu = 0$)

Independently draw $A_i \sim U(0, x_0)$ at time zero and stop on first exit from $(x_0 - A_i, x_0 + A_i)$. These strategies form an equilibrium.

Proof

Conditional on $A_i = a$, each endpoint is hit with probability $1/2$. Mixing over a makes $X_{\tau_i}^i$ uniform on $[0, 2x_0]$.

Against this distribution, the winning payoff is $v(x) = \min\{x/(2x_0), 1\}$ for $x \geq 0$. It is concave, so $v(X_{t \wedge \tau_0})$ is a bounded supermartingale.

Every deviation pays at most $v(x_0) = 1/2$; the proposed rule attains $1/2$.

Negative Drift and Risk Taking

For $\mu < 0$, define the scale function

$$s(x) = \frac{e^{-2\mu x/\sigma^2} - 1}{-2\mu/\sigma^2}, \quad \mathcal{L}s = 0.$$

Indifference gives $v(x) = s(x)/s(\bar{x})$ and $s(\bar{x}) = ns(x_0)$.

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For $n = 2$, draw $A \sim U(0, 1)$ and stop when $s(X_t)/s(x_0)$ exits $(1 - A, 1 + A)$.

Optional sampling gives equal endpoint probabilities; the construction induces G . The payoff $\min\{s(x)/(2s(x_0)), 1\}$ is a concave function of the stopped scale martingale, verifying the best response.

The winning payoff is convex on the interior support: rank incentives reward dispersion despite negative drift.

Private Incentives and Expected Losses

For $n = 2$ and $\mu < 0$, the constructed rule has bounded state support and finite expected duration.

Wald's identity gives

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The same tools now verify a best response to an endogenous payoff.

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Further foundations and precise versions of the probabilistic results.