Search in Continuous Time

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Topics in Economic Theory

Overview

1. Searching on a Brownian Motion

2. Maximum of Brownian Motion and Correlated Search

Searching on a Brownian Motion

Callander (2011 AER)

Motivation

Overview

1. Searching on a Brownian Motion

2. Maximum of Brownian Motion and Correlated Search
   - Correlated Search
   - Running Maximum of BM
   - Optimal Stopping
   - Optimal Correlated Search Intensity
   - Insights
   - Outcomes

Correlated Search

Motivation

Geological prospection, Drug development, Online browsing (Amazon, Netflix, Airbnb)

Multitude of options, ordered discovered

Correlated values: value of current items informative about value of items still unexplored

Search intensity: intensity increases when find something really good and just before stopping and taking best alternative (one last push kind of search)

Correlated Search

Setup (Urgun & Yariv (2023 JPE))

Sequence of alternatives: $T = \mathbb{R}_+$

Payoff: $u(t) = B_t$, std BM

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Speed: DM chooses speed of search $\sigma_t \in [\underline{\sigma}, \overline{\sigma}] \subset \mathbb{R}_{++}$

By time t, DM has gone through $\int_0^t \sigma_s \, ds$ alternatives

Note: If $\sigma_t = \sigma$, then $\int_0^t \sigma_s \, ds = \sigma t$

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Cost of Search Intensity: $c : [\underline{\sigma}, \overline{\sigma}] \to \mathbb{R}_+$, $\mathcal{C}^2$, increasing, log-convex (i.e. $c'(\boldsymbol{\sigma})/c(\boldsymbol{\sigma})$ increasing)

Terminal Payoff: $M_t := \max_{0 \leq s \leq t} X_s$ (recall)

Discounting: at rate $r > 0$

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Optimal Control and Stopping: $\sup_{\tau, \{\boldsymbol{\sigma}_t\}_t} \mathbb{E}\left[\exp(-r\tau) M_\tau - \int_0^\tau \exp(-rt) c(\boldsymbol{\sigma}_t) \, dt\right]$

Running Maximum of BM

Properties of M: $M_t := \max_{0 \le s \le t} B_s$

$M_t \ge 0$

For $a > 0$, let $\tau_a := \inf\{t \ge 0 \mid B_t = a\}$

By symmetry: $\mathbb{P}(B_t \ge a \mid \tau \le t) = \mathbb{P}(B_t > a \mid \tau \le t) = \mathbb{P}(B_t \le a \mid \tau \le t) = 1/2$

Reflection Principle: $\{M_t \ge a\} = \{\tau_a \le t\}$, hence

$$\begin{aligned}\mathbb{P}(M_t \ge a) &= \mathbb{P}(\tau_a \le t) = \mathbb{P}(\tau_a \le t \ \&\ B_t > a) + \mathbb{P}(\tau_a \le t \ \&\ B_t \le a) \\ &= 2\mathbb{P}(\tau_a \le t \ \&\ B_t \ge a) = 2\mathbb{P}(M_t \ge a \ \&\ B_t \ge a) \\ &= 2\mathbb{P}(B_t \ge a) = \mathbb{P}(B_t \ge a) + \mathbb{P}(B_t \le -a) \\ &= \mathbb{P}(|B_t| \ge a)\end{aligned}$$

Stopping the Correlated Search

Optimal Stopping $\tau := \inf\{t \geq 0 \mid g(M_t) \geq X_t\}$

Stop as soon as current value is low relative to maximum value thus far

Let $Z_t = (M_t, X_t)$; $\quad V(z) := \sup_{\tau, \{\sigma_t\}_t} \mathbb{E}\left[\exp(-r\tau)M_\tau - \int_0^\tau \exp(-rt)c(\sigma_t)\,\mathrm{d}t \mid Z_0 = z\right]$

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When $X_t = x < m = M_t$, $\mathrm{d}Z_t = \mathrm{d}(X_t, M_t) = (\mathrm{d}X_t, 0)$, implying...

HJB: for $x \in (g(m), m)$

$$rV(m,x) = \sup_\sigma -c(\sigma) + \frac{\sigma^2}{2} V_{xx}(m,x)$$

$$= -c(\sigma^r(m,x)) + \frac{\sigma^r(m,x)}{2} c'(\sigma^r(m,x))$$

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Value matching: $V(m, g(m)) = m$; $\quad$ **Smooth pasting**: $V_x(m, g(m)) = 0$

Letting $T_{[g(m),m]} = \inf\{t \geq 0 \mid X_t \notin (g(m), m)\}$ (1st exit time), we get for $x \in (g(m), m)$

$$V(m,x) = \mathbb{E}\left[V(X_{T_{[g(m),m]}}, M_{T_{[g(m),m]}})\right] - \mathbb{E}\left[\int_0^{T_{[g(m),m]}} \sigma^r(m, X_t)\,dt\right]$$

Stopping the Correlated Search

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Important observation: Letting $T_{[g(m),m]} = \inf\{t \geq 0 \mid X_t \notin (g(m), m)\}$ (1st exit time), we get for $x \in (g(m), m)$

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Fact: can get explicit functional form for $f(x) := \mathbb{E}\left[\int_0^{T_{[g(m),m]}} \sigma^r(m, X_t)\, dt\right]$

Important reference: **Borodin and Salminen (2002)**: book of formulas for Brownian motion and related processes

Putting it all together plus derivations...

Optimal Correlated Search Intensity

Proposition 1 (Urgun & Yariv (2023 JPE))

For g continuous, optimal search speed $\boldsymbol{\sigma}^r(m,x)$ for discounting $r \geq 0$, when interior, solves

$$2\mathbb{E}[\exp(-rT_{[g(m),m]}) \mid X_0 = x] = \boldsymbol{\sigma}^r(m,x)\frac{c'(\boldsymbol{\sigma}^r(m,x))}{c(\boldsymbol{\sigma}^r(m,x))} = \text{elasticity of } c \text{ at } \boldsymbol{\sigma}^r(m,x)$$

Corollary 1 (Urgun & Yariv (2023 JPE))

Since $\mathbb{E}[\exp(-rT_{[g(m),m]}) \mid X_0 = x]$ is symmetric around and minimized at $(g(m)+m)/2$, and decreasing in r, then $\boldsymbol{\sigma}^r(m,x)$ is too.

Furthermore, $\boldsymbol{\sigma}^r(m,m) = \boldsymbol{\sigma}^r(m,g(m)) = \boldsymbol{\sigma}^0$ for any $r > 0$ and m, where, when interior, $\boldsymbol{\sigma}^0$ solves $\boldsymbol{\sigma}^0 \frac{c'(\boldsymbol{\sigma}^0)}{c(\boldsymbol{\sigma}^0)} = 2$

(**Key steps in proof**)

Optimal Correlated Search Intensity

Proposition 2 (Urgun & Yariv (2023 JPE))

The stopping boundary is given by $g^r(m) = m - d^r$ where d^r solves

$$d^r \left(1 + 2\sinh(\sqrt{r/2}d^r)\tanh(\sqrt{r/2}d^r)\right) = \frac{(\sigma^0)^2}{2c(\sigma^0)} =: d^0$$

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Intuition: Continue searching until you get realizations that are low enough relative to current maximum, i.e. stop when $X_t \leq M_t - d^r$

Optimal Correlated Search Intensity

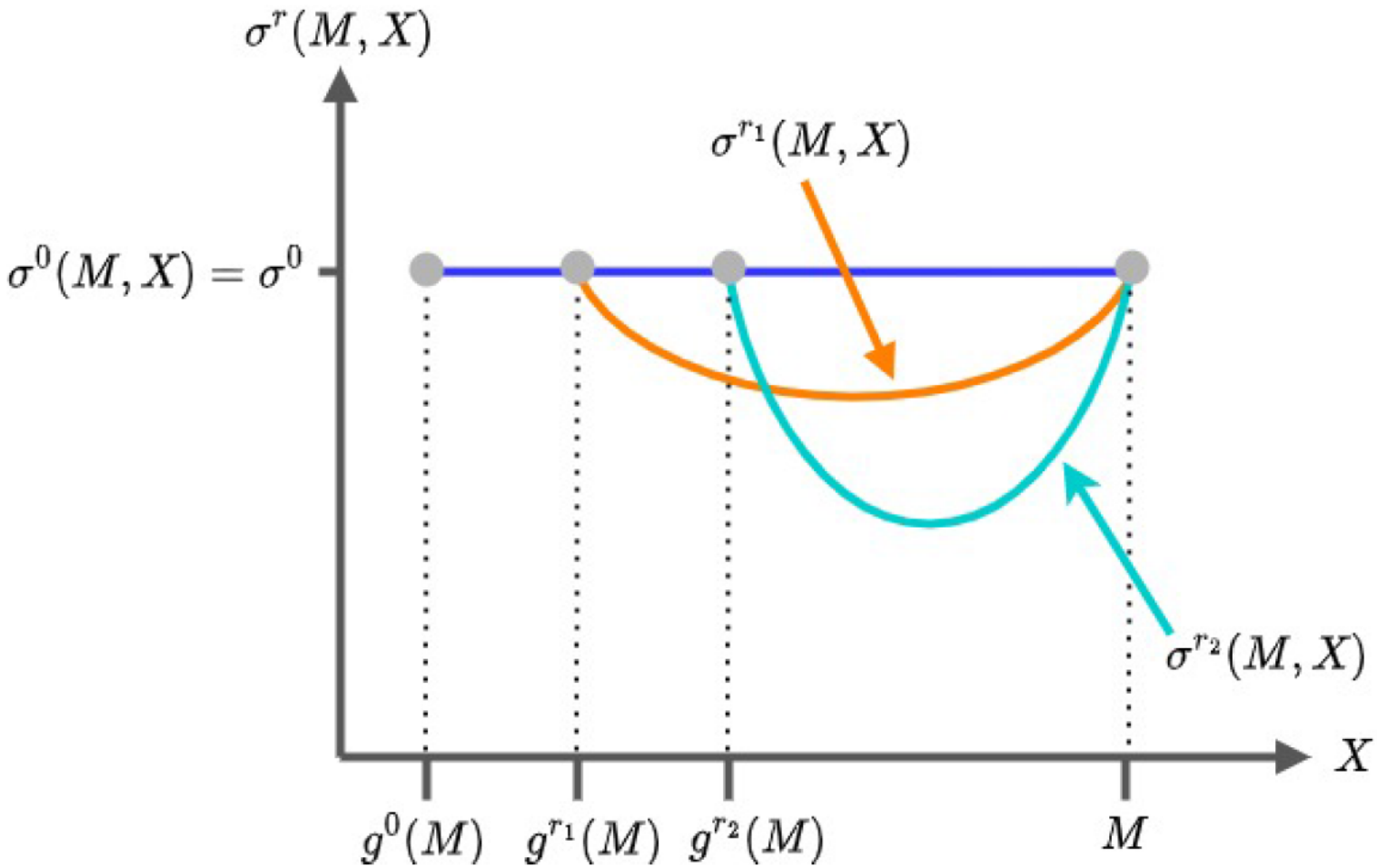


FIGURE 2: Features of the optimal search policy with $r_2 > r_1 > 0$

Correlated Search: Insights

Motivation

Correlated values: value of current items informative about value of items still unexplored

Search intensity: intensity increases when find something really good and just before stopping and taking best alternative (one last push kind of search)

Outcomes of Correlated Search

Expected Search Outcome: $\mathbb{E}[M_\tau] = \mathbb{E}[M_\tau - X_\tau] = d^r$

Value: $V(0,0) = \mathbb{E}[\exp(-r\tau)M_\tau - \int_0^\tau \exp(-rt)c(\sigma(M_t, X_t))\,\mathrm{d}t] = \frac{d^r}{d^0}\frac{\sinh(\sqrt{r/2}d^r)}{\sqrt{2r}}$

No-Discounting Case:

Expected Search Time: $\mathbb{E}[\tau] = (V(0,0) - \mathbb{E}[M_\tau])/c(\sigma^0) = \frac{d^0}{2c(\sigma^0)}$

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Proposition 1 (Urgun & Yariv (2023 JPE)) Proof sketch (Back)

- **Fact**: (see paper for references)

$$\mathbb{E}\left[\int_0^{T_{[g(m),m]}} \sigma^r(X_t, m)\,\mathrm{d}t\right] = f(x) := \int_{g(m)}^{m} G^r_{g(m),m}(x,y)\,\frac{2c(\sigma^r(m,y))}{\sigma^r m, y^2}\,\mathrm{d}y$$

where $G^r_{a,b}(x,y)$ is the Green's function of a BM with exponential 'killing rate', where $a < x, y < b$

$$G^r_{a,b}(x,y) = \frac{\sinh(\sqrt{2r}(b - x \vee y))\sinh(\sqrt{2r}(x \wedge y - a))}{\sqrt{2r}\sinh(\sqrt{2r}(b-a))}$$

Note: $f(x) \to 0$ as $x \downarrow g(m)$

Proposition 1 (Urgun & Yariv (2023 JPE)) Proof sketch (Back)

- **Breaking into small pieces**:

$$\mathbb{E}\left[V(X_{T_{[g(m),m]}}, M_{T_{[g(m),m]}})\right] = \mathbb{E}\left[V(X_{T_{[g(m),m]}}, m)\right] = p(x)m + (1 - p(x))V(m,m)$$

where $V(m, g(m)) = m$ (value matching) and $p(x) = \mathbb{P}(X_{T_{[g(m),m]}} = g(m))$

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  where $V(m, g(m)) = m$ (value matching) and $p(x) = \mathbb{P}(X_{T_{[g(m),m]}} = g(m))$

- X_t is a martingale $\implies$ $X_{T_{[g(m),m]}}$ is a martingale $\implies$ $x = \mathbb{E}[X_0] = p(x)g(m) + (1 - p(x))m \iff p(x) = \frac{m-x}{m-g(m)}$

Proposition 1 (Urgun & Yariv (2023 JPE)) Proof sketch (Back)

- **Putting it together**: $\forall x \in (g(m), m)$

$$V(m,x) = p(x)m + (1-p(x))V(m,m) - f(x)$$

$$\iff V(m,x) - m = -(1-p(x))m + (1-p(x))V(m,m) - f(x)$$

$$\iff V(m,m) - m = \frac{V(m,x) - m}{1-p(x)} + \frac{f(x)}{1-p(x)}$$

$$= \frac{V(m,x) - m}{x - g(m)}(m - g(m)) + \frac{f(x)}{x - g(m)}(m - g(m))$$

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- As $V(m,x) \to m$ as $x \downarrow g(m)$, we get $\frac{V(m,x)-m}{x-g(m)} \to V_x(m, g(m)) = 0$ by smooth pasting

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- As $V(m,x) \to m$ as $x \downarrow g(m)$, we get $\frac{V(m,x)-m}{x-g(m)} \to V_x(m,g(m)) = 0$ by smooth pasting
- Since $f(x) \to 0$ as $x - g(m) \to 0$, use l'Hopital's rule:

$$(m-g(m)) \lim_{x \downarrow g(m)} \frac{f(x)}{x-g(m)}$$

$$= (m-g(m)) \int_{g(m)}^{m} \frac{\sinh(\sqrt{2r}(m-y))}{\sinh(\sqrt{2r}(m-g(m)))} \frac{2c(\sigma^r(m,y))}{\sigma^r m, y^2} \, dy =: K$$

Proposition 1 (Urgun & Yariv (2023 JPE)) Proof sketch (Back)

- **Putting it together again**:

$$V(m,m) = m + K(m - g(m))$$

$$\implies V(m,x) = m + (1 - p(x))K(m - g(m)) - f(x)$$

- Use closed form for $V(m,x)$ together with optimality condition from HJB to get,

$$c'(\sigma^r(m,x))\sigma^r(m,x)/2 = V_{xx}(m,x) = -p''(x)K(m - g(m)) - f''(x) = -f''(x)$$

$$\implies \int_{g(m)}^{m} c'(\sigma^r(m,x))\sigma^r(m,x)/2 + f''(x)\,\mathrm{d}x = 0$$

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- After some squinting and manipulations, the result obtains

Outcomes of Correlated Search

Proposition 3 (Urgun & Yariv (2023 JPE)) Proof sketch

- **Value Function**:

$$
\begin{aligned}
V(m,m) &= m + K(m - g(m)) \\
&= m + \left((m - g(m)) \int_{g(m)}^{m} \frac{\sinh(\sqrt{2r}(m-y))}{\sinh(\sqrt{2r}(m-g(m)))} \frac{2c(\sigma^r(m,y))}{\sigma^r m,y^2} \, dy \right) (m - g(m)) \\
&= (d^r)^2 \left(\int_{g(m)}^{m} \frac{\sinh(\sqrt{2r}(m-y))}{\sinh(\sqrt{2r}d^r)} \frac{2c(\sigma^r(m,y))}{\sigma^r m,y^2} \, dy \right)
\end{aligned}
$$

- Result follows after some straightforward algebra