

# Stopping in Continuous Time

When should an economic agent act?

Duarte Gonçalves

University College London

Topics in Economic Theory

5 October 2026

# To Buy or Not to Buy?

## **Branco, Sun, & Villas-Boas (2012 MnSc)**

A consumer considers buying a product.

Read another review, try it in a shop, or ask someone who has one?

Each reveals something about whether the product is right for the consumer.

# To Buy or Not to Buy?

## Branco, Sun, & Villas-Boas (2012 MnSc)

A consumer considers buying a product.

Read another review, try it in a shop, or ask someone who has one?

Each reveals something about whether the product is right for the consumer.

Expected valuation minus price:  $x$ . Outside option:  $0$ .

Stop now: buy if  $x \geq 0$ , walk away otherwise. **Scrap value:**

$$v(x) := \max\{x, 0\} = x^+.$$

Continue: pay flow cost  $c > 0$  to acquire more information.

Is positive expected net utility enough to justify buying?

## Learning about Attributes

Product has many attributes: each is good or bad for the consumer.

One new attribute every  $\Delta$  units of time; independent  $\xi_k \in \{-1, 1\}$  with equal probability.

$$X_t^\Delta = x + \sigma\sqrt{\Delta} \sum_{k=1}^{\lfloor t/\Delta \rfloor} \xi_k.$$

## Learning about Attributes

Product has many attributes: each is good or bad for the consumer.

One new attribute every  $\Delta$  units of time; independent  $\xi_k \in \{-1, 1\}$  with equal probability.

$$X_t^\Delta = x + \sigma\sqrt{\Delta} \sum_{k=1}^{\lfloor t/\Delta \rfloor} \xi_k.$$

A **stochastic process**: a random variable at each date, a path of discoveries for each realisation.

At grid times, an additional  $h$  units of search gives

$$\mathbb{E}[X_{t+h}^\Delta - X_t^\Delta] = 0, \quad \text{Var}(X_{t+h}^\Delta - X_t^\Delta) = \sigma^2 h.$$

No expected change in the valuation; more information about what to do.

## Brownian Information

Let the time per attribute shrink, keeping variance per unit time fixed.

The interpolated random walk converges in distribution to

$$X_t = x + \sigma B_t.$$

### Definition (Brownian Motion)

$B_0 = 0$ ; paths are continuous a.s.; for  $s < t$ ,  $B_t - B_s \sim N(0, t - s)$  independently of information available at  $s$ .

A small interval  $h$  reveals a mean-zero innovation of size  $\sqrt{h}$ .

$\sigma$ : how much product information is revealed per unit time.

Stationary approximation: we abstract from exhausting a finite list of attributes.

## Why Learn If Expected Utility Does Not Change?

Suppose the consumer is initially indifferent:  $x = 0$ .

Search for a fixed time  $h$ , then buy iff  $X_h \geq 0$ .

$$\mathbb{E}_0[v(X_h) - ch] = \sigma \sqrt{\frac{h}{2\pi}} - ch.$$

## Why Learn If Expected Utility Does Not Change?

Suppose the consumer is initially indifferent:  $x = 0$ .

Search for a fixed time  $h$ , then buy iff  $X_h \geq 0$ .

$$\mathbb{E}_0[v(X_h) - ch] = \sigma \sqrt{\frac{h}{2\pi}} - ch.$$

**For small  $h$ , searching is strictly better than stopping.**

Information is valuable because good news can induce a purchase; bad news can induce rejection.

Near indifference, the benefit is of order  $\sqrt{h}$ ; the cost is of order  $h$ .

## Why Learn If Expected Utility Does Not Change?

Suppose the consumer is initially indifferent:  $x = 0$ .

Search for a fixed time  $h$ , then buy iff  $X_h \geq 0$ .

$$\mathbb{E}_0[v(X_h) - ch] = \sigma \sqrt{\frac{h}{2\pi}} - ch.$$

**For small  $h$ , searching is strictly better than stopping.**

Information is valuable because good news can induce a purchase; bad news can induce rejection.

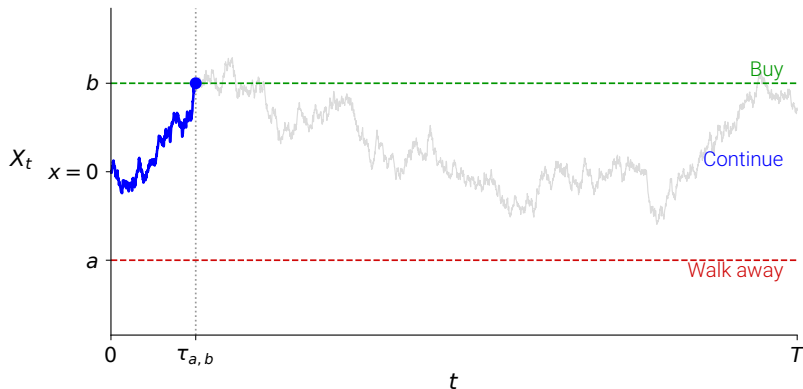
Near indifference, the benefit is of order  $\sqrt{h}$ ; the cost is of order  $h$ .

Best fixed search duration:

$$h^* = \frac{\sigma^2}{8\pi c^2}, \quad \mathbb{E}_0[v(X_{h^*}) - ch^*] = \frac{\sigma^2}{8\pi c}.$$

But why commit to the same duration after good and bad reviews?

## Responding to the Information



Good enough news: buy. Bad enough news: walk away.

Between the boundaries: keep learning.

Choose a rule that responds to discoveries, rather than a date in advance.

# Overview

1. Search for Product Information
2. Itô Calculus: The Value of Further Information
3. Dynamic Programming and Free Boundaries
4. Investment: The Value of Waiting
5. Entry and Exit: Hysteresis
6. Strategic Stopping: Races and Contests

## What Can the Consumer Condition On?

Information after observing reviews up to  $t$ :

$$\mathcal{F}_t^X := \sigma(X_s : 0 \leq s \leq t).$$

The family  $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$  is a **filtration**:  $\mathcal{F}_s \subseteq \mathcal{F}_t$  for  $s \leq t$ .

$X$  is **adapted**:  $X_t$  is  $\mathcal{F}_t$ -measurable.

# What Can the Consumer Condition On?

Information after observing reviews up to  $t$ :

$$\mathcal{F}_t^X := \sigma(X_s : 0 \leq s \leq t).$$

The family  $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$  is a **filtration**:  $\mathcal{F}_s \subseteq \mathcal{F}_t$  for  $s \leq t$ .

$X$  is **adapted**:  $X_t$  is  $\mathcal{F}_t$ -measurable.

The consumer can react to a review already read, not to tomorrow's review.

We use the natural filtration, completed and made right-continuous.

Independence of Brownian increments means future news is independent of  $\mathcal{F}_t$ .

# Choosing When to Stop

## Definition (Stopping Time)

$\tau : \Omega \rightarrow [0, \infty]$  is a stopping time if  $\{\tau \leq t\} \in \mathcal{F}_t$  for every  $t \geq 0$ . By time  $t$ , the consumer knows whether stopping has occurred.

Feasible: stop at a fixed date; stop at the first sufficiently good review; stop when  $X$  exits  $(a, b)$ .

Infeasible: stop when  $X$  reaches its highest value over the next hour.

# Choosing When to Stop

## Definition (Stopping Time)

$\tau : \Omega \rightarrow [0, \infty]$  is a stopping time if  $\{\tau \leq t\} \in \mathcal{F}_t$  for every  $t \geq 0$ . By time  $t$ , the consumer knows whether stopping has occurred.

Feasible: stop at a fixed date; stop at the first sufficiently good review; stop when  $X$  exits  $(a, b)$ .

Infeasible: stop when  $X$  reaches its highest value over the next hour.

Consumer's value:

$$V(x) := \sup_{\tau \in \mathbb{T}_1} \mathbb{E}_x[v(X_\tau) - c\tau], \quad \mathbb{T}_1 := \{\tau \text{ stopping time} : \mathbb{E}\tau < \infty\}.$$

$V(x) \geq v(x)$ : immediate stopping is feasible.

## Which Part of the History Matters?

Given current expected net utility  $X_t$ , future news satisfies

$$X_{t+h} = X_t + \sigma(B_{t+h} - B_t), \quad B_{t+h} - B_t \perp\!\!\!\perp \mathcal{F}_t.$$

**Markov property**: two review histories ending at the same  $X_t$  give the same future problem.

## Which Part of the History Matters?

Given current expected net utility  $X_t$ , future news satisfies

$$X_{t+h} = X_t + \sigma(B_{t+h} - B_t), \quad B_{t+h} - B_t \perp\!\!\!\perp \mathcal{F}_t.$$

**Markov property:** two review histories ending at the same  $X_t$  give the same future problem.

Search costs already paid are sunk. No deadline; no change in the information technology.

Thus continuation value depends on  $X_t$ , not on elapsed time or the route taken to get there.

If reviews become less informative as attributes are exhausted, this state description must change.

# Valuation Is a Martingale; the Scrap Value Is Not

## Definition (Martingale)

An adapted, integrable process  $M$  is a martingale if  $\mathbb{E}[M_t \mid \mathcal{F}_s] = M_s$  for  $s \leq t$ . Replace  $=$  by  $\geq$  for a submartingale, or  $\leq$  for a supermartingale.

$X_t = x + \sigma B_t$  is a martingale: another review does not predictably raise valuation.

# Valuation Is a Martingale; the Scrap Value Is Not

## Definition (Martingale)

An adapted, integrable process  $M$  is a martingale if  $\mathbb{E}[M_t \mid \mathcal{F}_s] = M_s$  for  $s \leq t$ . Replace  $=$  by  $\geq$  for a submartingale, or  $\leq$  for a supermartingale.

$X_t = x + \sigma B_t$  is a martingale: another review does not predictably raise valuation.

But the consumer may change the purchase decision:

$$\mathbb{E}[v(X_t) \mid \mathcal{F}_s] \geq v(\mathbb{E}[X_t \mid \mathcal{F}_s]) = v(X_s).$$

Convexity of  $v$  gives the option value of information.

Need to compare that gain with the cost of waiting.

## Evaluating a Purchase Rule

Fix  $a < 0 < b$  and  $a < x < b$ . Consider

$$\tau_{a,b} := \inf\{t \geq 0 : X_t \notin (a,b)\}.$$

At  $b$ , buy and get  $b$ ; at  $a$ , walk away and get  $0$ .

## Evaluating a Purchase Rule

Fix  $a < 0 < b$  and  $a < x < b$ . Consider

$$\tau_{a,b} := \inf\{t \geq 0 : X_t \notin (a,b)\}.$$

At  $b$ , buy and get  $b$ ; at  $a$ , walk away and get  $0$ .

Expected payoff under this rule:

$$W_{a,b}(x) = b \mathbb{P}_x(X_{\tau_{a,b}} = b) - c \mathbb{E}_x[\tau_{a,b}].$$

Need purchase probability and expected search duration.

The martingale property can deliver both, if it survives stopping at a random time.

## Can Favourable Timing Raise Expected Valuation?

### Theorem (Optional Sampling)

For a càdlàg martingale  $M$  and bounded stopping times  $S \leq T$ ,  $\mathbb{E}[M_T \mid \mathcal{F}_S] = M_S$ . The same holds for a.s. finite times if  $(M_{t \wedge T})_{t \geq 0}$  is uniformly integrable; boundedness of that stopped process is sufficient.

Under the purchase rule,  $X_{t \wedge \tau_{a,b}} \in [a, b]$ .

Brownian motion exits a bounded interval a.s.; optional sampling gives

$$\mathbb{E}_x[X_{\tau_{a,b}}] = x.$$

## Can Favourable Timing Raise Expected Valuation?

### Theorem (Optional Sampling)

For a càdlàg martingale  $M$  and bounded stopping times  $S \leq T$ ,  $\mathbb{E}[M_T \mid \mathcal{F}_S] = M_S$ . The same holds for a.s. finite times if  $(M_{t \wedge T})_{t \geq 0}$  is uniformly integrable; boundedness of that stopped process is sufficient.

Under the purchase rule,  $X_{t \wedge \tau_{a,b}} \in [a, b]$ .

Brownian motion exits a bounded interval a.s.; optional sampling gives

$$\mathbb{E}_x[X_{\tau_{a,b}}] = x.$$

Choosing when to stop does not raise expected valuation here.

It raises the expected payoff by selecting *whether to buy*.

## Purchase Probability

Continuous paths imply  $X_{\tau_{a,b}} \in \{a, b\}$ .

Write  $p(x) = \mathbb{P}_x(X_{\tau_{a,b}} = b)$ . Then

$$x = p(x)b + (1 - p(x))a \implies p(x) = \frac{x - a}{b - a}.$$

## Purchase Probability

Continuous paths imply  $X_{\tau_{a,b}} \in \{a, b\}$ .

Write  $p(x) = \mathbb{P}_x(X_{\tau_{a,b}} = b)$ . Then

$$x = p(x)b + (1 - p(x))a \implies p(x) = \frac{x - a}{b - a}.$$

Higher initial expected net utility raises the probability of purchase.

Higher purchase threshold  $b$  lowers it: more favourable news is needed.

A lower rejection threshold  $a$  raises it: more chances to recover after bad news.

## Purchase Probability

Continuous paths imply  $X_{\tau_{a,b}} \in \{a, b\}$ .

Write  $p(x) = \mathbb{P}_x(X_{\tau_{a,b}} = b)$ . Then

$$x = p(x)b + (1 - p(x))a \implies p(x) = \frac{x - a}{b - a}.$$

Higher initial expected net utility raises the probability of purchase.

Higher purchase threshold  $b$  lowers it: more favourable news is needed.

A lower rejection threshold  $a$  raises it: more chances to recover after bad news.

No  $\sigma$ ? For fixed boundaries, volatility changes the speed at which news arrives, not which boundary is reached first in distribution.

## Expected Search Duration

Independent Brownian increments imply  $\mathbb{E}[B_t^2 - t \mid \mathcal{F}_s] = B_s^2 - s$ .

Thus  $(X_t - x)^2 - \sigma^2 t$  is a martingale.

Stop it at  $\tau_{a,b} \wedge T$  and let  $T \rightarrow \infty$ :

$$\sigma^2 \mathbb{E}_x \tau_{a,b} = \mathbb{E}_x[(X_{\tau_{a,b}} - x)^2] = (x - a)(b - x).$$

## Expected Search Duration

Independent Brownian increments imply  $\mathbb{E}[B_t^2 - t \mid \mathcal{F}_s] = B_s^2 - s$ .

Thus  $(X_t - x)^2 - \sigma^2 t$  is a martingale.

Stop it at  $\tau_{a,b} \wedge T$  and let  $T \rightarrow \infty$ :

$$\sigma^2 \mathbb{E}_x \tau_{a,b} = \mathbb{E}_x[(X_{\tau_{a,b}} - x)^2] = (x - a)(b - x).$$

Expected search is longest in the middle; it is short near either decision boundary.

For fixed boundaries, faster information arrival reduces duration in proportion to  $1/\sigma^2$ .

The stopped square is bounded; this also proves  $\mathbb{E} \tau_{a,b} < \infty$ .

[► Details](#)

## How Much Evidence Should the Consumer Require?

Start at  $x = 0$  and consider symmetric thresholds  $(-b, b)$ .

Purchase with probability  $1/2$ ; expected search duration  $b^2/\sigma^2$ .

$$W_{-b,b}(0) = \frac{b}{2} - \frac{cb^2}{\sigma^2}.$$

## How Much Evidence Should the Consumer Require?

Start at  $x = 0$  and consider symmetric thresholds  $(-b, b)$ .

Purchase with probability  $1/2$ ; expected search duration  $b^2/\sigma^2$ .

$$W_{-b,b}(0) = \frac{b}{2} - \frac{cb^2}{\sigma^2}.$$

Best threshold within this class:

$$b^* = \frac{\sigma^2}{4c}, \quad W_{-b^*,b^*}(0) = \frac{\sigma^2}{16c}.$$

More costly search: require less evidence before deciding.

More informative reviews: it is worth waiting for stronger evidence.

## How Much Evidence Should the Consumer Require?

Start at  $x = 0$  and consider symmetric thresholds  $(-b, b)$ .

Purchase with probability  $1/2$ ; expected search duration  $b^2/\sigma^2$ .

$$W_{-b,b}(0) = \frac{b}{2} - \frac{cb^2}{\sigma^2}.$$

Best threshold within this class:

$$b^* = \frac{\sigma^2}{4c}, \quad W_{-b^*,b^*}(0) = \frac{\sigma^2}{16c}.$$

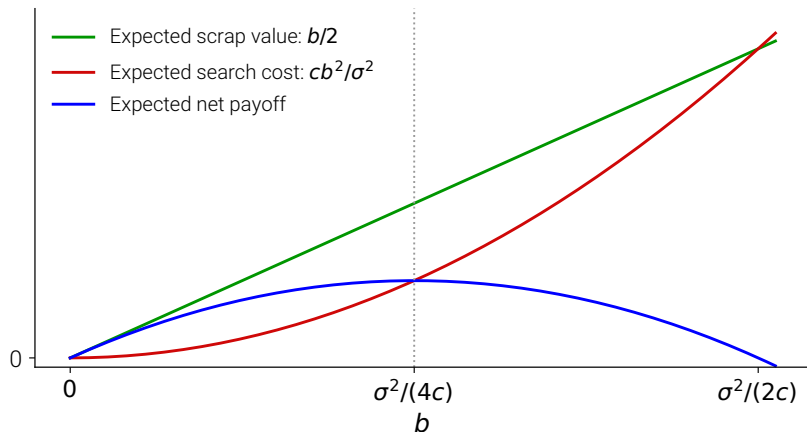
More costly search: require less evidence before deciding.

More informative reviews: it is worth waiting for stronger evidence.

This rule improves on the best fixed date by a factor  $\pi/2$ .

We have not yet ruled out asymmetric or history-dependent rules.

# The Cost and Benefit of More Evidence



Raising  $b$  increases the payoff conditional on purchase linearly.

It increases expected time, and hence expected cost, quadratically.

The best threshold equates these marginal effects.

## Faster Information Can Mean Longer Search

With the best symmetric rule, starting at 0,

$$b^* = \frac{\sigma^2}{4c}, \quad \mathbb{E}_0 \tau_{-b^*, b^*} = \frac{\sigma^2}{16c^2}.$$

Higher  $\sigma$  speeds up the same path distribution, holding the thresholds fixed.

## Faster Information Can Mean Longer Search

With the best symmetric rule, starting at 0,

$$b^* = \frac{\sigma^2}{4c}, \quad \mathbb{E}_0 \tau_{-b^*, b^*} = \frac{\sigma^2}{16c^2}.$$

Higher  $\sigma$  speeds up the same path distribution, holding the thresholds fixed.

But it also makes additional information more valuable, so the consumer raises the thresholds.

Here the policy response dominates: faster information arrival produces *longer* search.

To predict response times, the stopping rule must be endogenous.

## Is This Rule Actually Optimal?

Set  $b = \sigma^2/(4c)$ . The same exit calculations give its payoff at every  $x$ :

$$\tilde{V}(x) = \begin{cases} 0, & x \leq -b, \\ \frac{c}{\sigma^2}(x+b)^2, & -b < x < b, \\ x, & x \geq b. \end{cases}$$

Therefore  $V \geq \tilde{V} \geq v$ : this is a feasible rule.

## Is This Rule Actually Optimal?

Set  $b = \sigma^2/(4c)$ . The same exit calculations give its payoff at every  $x$ :

$$\tilde{V}(x) = \begin{cases} 0, & x \leq -b, \\ \frac{c}{\sigma^2}(x+b)^2, & -b < x < b, \\ x, & x \geq b. \end{cases}$$

Therefore  $V \geq \tilde{V} \geq v$ : this is a feasible rule.

To prove optimality, need the reverse inequality  $V \leq \tilde{V}$ .

Can any stopping time yield more than  $\tilde{V}(x)$ ?

Need to understand how  $\tilde{V}(X_t) - ct$  evolves as reviews arrive.

# Overview

1. Search for Product Information
2. Itô Calculus: The Value of Further Information
3. Dynamic Programming and Free Boundaries
4. Investment: The Value of Waiting
5. Entry and Exit: Hysteresis
6. Strategic Stopping: Races and Contests

## How Does Another Review Change Continuation Value?

Inside the proposed continuation region,  $\Delta X = \sigma \Delta B$  and

$$\Delta \tilde{V} \approx \tilde{V}'(x) \sigma \Delta B + \frac{1}{2} \tilde{V}''(x) \sigma^2 (\Delta B)^2.$$

First-order effect: good or bad news, depending on the review.

Second-order effect: the flexibility to respond to news is valuable.

## How Does Another Review Change Continuation Value?

Inside the proposed continuation region,  $\Delta X = \sigma \Delta B$  and

$$\Delta \tilde{V} \approx \tilde{V}'(x) \sigma \Delta B + \frac{1}{2} \tilde{V}''(x) \sigma^2 (\Delta B)^2.$$

First-order effect: good or bad news, depending on the review.

Second-order effect: the flexibility to respond to news is valuable.

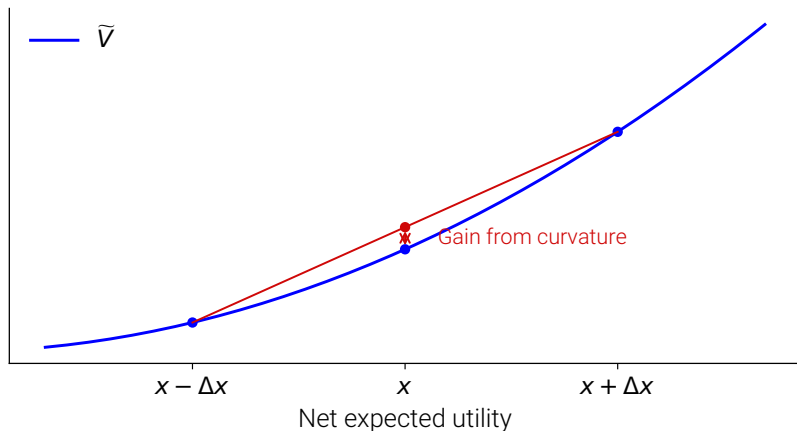
Since  $\mathbb{E} \Delta B = 0$  and  $\mathbb{E} (\Delta B)^2 = \Delta t$ ,

$$\mathbb{E} [\Delta \tilde{V} - c \Delta t] \approx \left( \frac{1}{2} \sigma^2 \tilde{V}''(x) - c \right) \Delta t.$$

The candidate has  $\tilde{V}'' = 2c/\sigma^2$ : marginal benefit equals marginal cost.

Itô calculus makes this argument exact, including at random stopping times.

## Curvature Is the Value of Flexibility

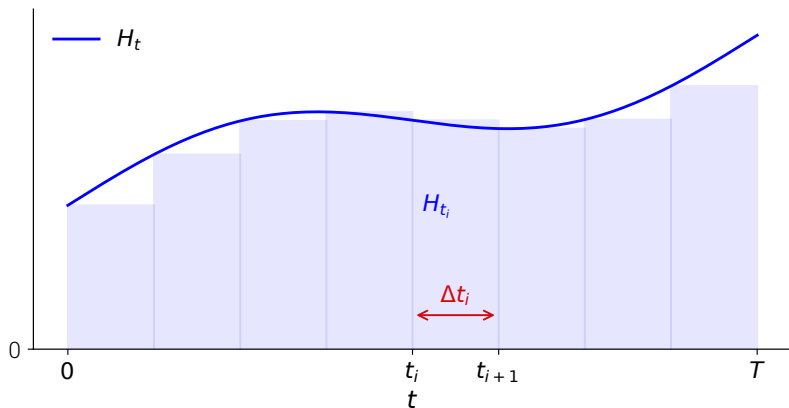


A mean-zero change in valuation can raise expected continuation value.

Locally, the gain is governed by curvature and the variance of the news.

We must retain the second-order term when adding up many small reviews.

## Ordinary Integration

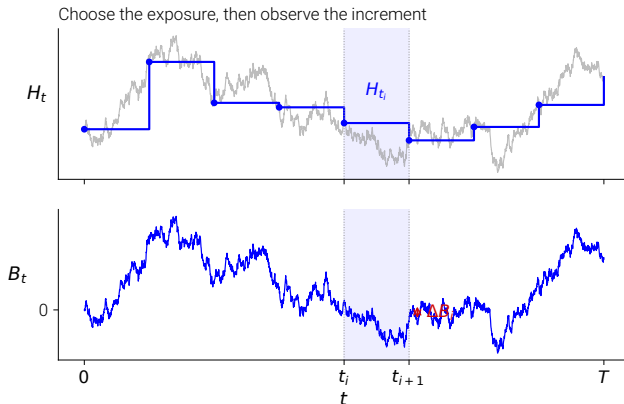


Approximate  $\int_0^T H_t dt$  by  $\sum_i H_{t_i} \Delta t_i$ .

Multiply a height by a time increment; add the rectangle areas.

Search costs are of this kind:  $\int_0^\tau c dt = c\tau$ .

# Integrating against News



Now add  $\sum_i H_{t_i}(B_{t_{i+1}} - B_{t_i})$ .

$H_{t_i}$  is fixed before the next review; the increment may have either sign.

In the consumer problem,  $H_t = \sigma \tilde{V}'(X_t)$  is value's sensitivity to news.

## The Integrand Cannot Anticipate the Review

For a simple process,

$$H_s = \sum_i H_i \mathbf{1}_{\{t_i < s \leq t_{i+1}\}}, \quad H_i \text{ is } \mathcal{F}_{t_i}\text{-measurable,}$$

define

$$\int_0^t H_s \, dB_s := \sum_i H_i (B_{t_{i+1} \wedge t} - B_{t_i \wedge t}).$$

## The Integrand Cannot Anticipate the Review

For a simple process,

$$H_s = \sum_i H_i \mathbf{1}_{\{t_i < s \leq t_{i+1}\}}, \quad H_i \text{ is } \mathcal{F}_{t_i}\text{-measurable,}$$

define

$$\int_0^t H_s \, dB_s := \sum_i H_i (B_{t_{i+1} \wedge t} - B_{t_i \wedge t}).$$

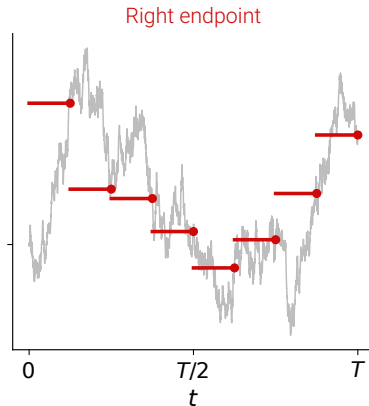
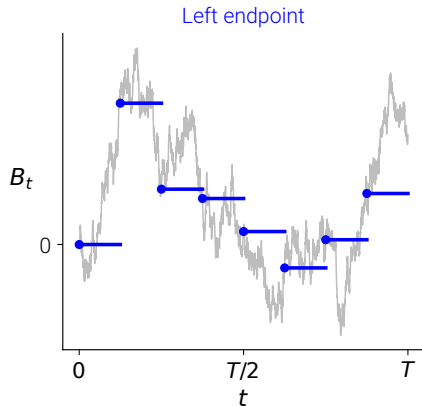
$H_i$  can depend on all past reviews, but not on the next innovation.

**Predictable processes:** measurable wrt the  $\sigma$ -field generated by left-continuous adapted processes.

Continuous adapted integrands, such as  $\tilde{V}'(X_t)$  here, are predictable.

This is the same information restriction as in choosing a feasible stopping rule.

# Why the Endpoint Matters

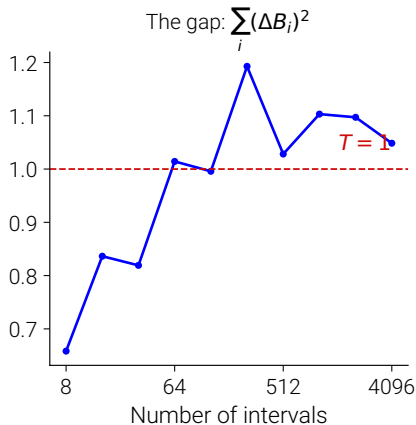
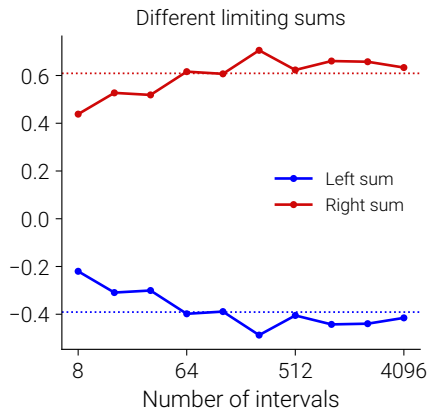


Take  $H = B$ . The two sums differ by

$$\sum_i B_{t_{i+1}} \Delta B_i - \sum_i B_{t_i} \Delta B_i = \sum_i (\Delta B_i)^2.$$

Using the right endpoint lets the weight respond to the very increment it multiplies.

# The Difference Does Not Vanish



Refining the partition does not reconcile left and right endpoint sums.

Brownian quadratic variation is  $[B]_T = T$ .

The ordinary rule  $\int B \, dB = B_T^2/2$  misses precisely this effect.

### Proposition (Brownian Quadratic Variation)

For deterministic partitions of  $[0, T]$  with mesh  $|\pi| \rightarrow 0$ ,  $\sum_i (\Delta B_i)^2 \rightarrow T$  in  $L^2$ .

### Proof

The sum has expectation  $T$ . Independence of increments gives

$$\text{Var} \left( \sum_i (\Delta B_i)^2 \right) = 2 \sum_i (\Delta t_i)^2 \leq 2T|\pi| \rightarrow 0.$$

Continuous finite-variation paths have zero quadratic variation.

Brownian paths have infinite total variation a.s.; ordinary Riemann–Stieltjes integration cannot cover general adapted integrands.

For the consumer's valuation,  $[X]_t = \sigma^2 t$ : accumulated variance of the news.

## Proposition

For a simple predictable  $H$  with  $\mathbb{E} \int_0^T H_s^2 ds < \infty$ ,

$$\mathbb{E} \left[ \left( \int_0^T H_s dB_s \right)^2 \right] = \mathbb{E} \int_0^T H_s^2 ds.$$

## Proof

Write the integral as  $\sum_i H_i \Delta B_i$ . Cross terms have zero expectation: condition on the information before the later increment. For each diagonal term,

$$\mathbb{E}[H_i^2 (\Delta B_i)^2] = \mathbb{E}[H_i^2] \Delta t_i.$$

Unpredictable innovations do not systematically reinforce one another.

Variance accumulates at rate  $H_t^2$ : more sensitivity to reviews means more uncertainty in continuation value.

## Definition (Itô Integral)

Let  $H$  be predictable with  $\mathbb{E} \int_0^T H_s^2 ds < \infty$ . Choose simple predictable  $H^n$  such that  $\mathbb{E} \int_0^T (H_s^n - H_s)^2 ds \rightarrow 0$ . Define

$$\int_0^T H_s dB_s := L^2\text{-}\lim_{n \rightarrow \infty} \int_0^T H_s^n dB_s.$$

The isometry makes the integrals Cauchy in  $L^2$ .

It also makes the limit independent of the approximating sequence.

## From Sums to an Integral

### Definition (Itô Integral)

Let  $H$  be predictable with  $\mathbb{E} \int_0^T H_s^2 ds < \infty$ . Choose simple predictable  $H^n$  such that  $\mathbb{E} \int_0^T (H_s^n - H_s)^2 ds \rightarrow 0$ . Define

$$\int_0^T H_s dB_s := L^2\text{-}\lim_{n \rightarrow \infty} \int_0^T H_s^n dB_s.$$

The isometry makes the integrals Cauchy in  $L^2$ .

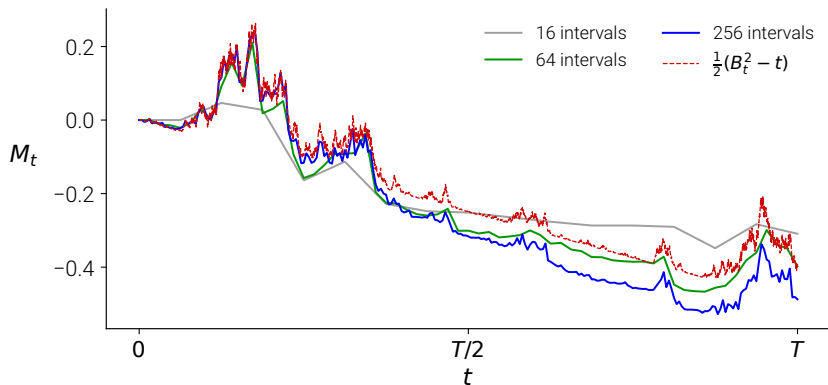
It also makes the limit independent of the approximating sequence.

Continuous bounded  $H$ : left-endpoint approximations work as the mesh shrinks.

General predictable  $H$ : use simple approximation in this norm; arbitrary point samples need not work.

The resulting integral process has a continuous version.

## Refining the Approximation



Here  $H_t = B_t$ . Refine the partition along the same realised path.

The limiting integral is  $\int_0^t B_s dB_s = \frac{1}{2}(B_t^2 - t)$ .

The correction  $-t/2$  removes the predictable accumulation of squared innovations.

## Properties: Linearity and Accumulation

For square-integrable predictable  $H, K$  and constants  $a, b$ ,

$$\int_0^t (aH_s + bK_s) dB_s = a \int_0^t H_s dB_s + b \int_0^t K_s dB_s.$$

Contributions add across time:

$$\int_0^t H_s dB_s = \int_0^u H_s dB_s + \int_u^t H_s dB_s, \quad 0 \leq u \leq t.$$

## Properties: Linearity and Accumulation

For square-integrable predictable  $H, K$  and constants  $a, b$ ,

$$\int_0^t (aH_s + bK_s) dB_s = a \int_0^t H_s dB_s + b \int_0^t K_s dB_s.$$

Contributions add across time:

$$\int_0^t H_s dB_s = \int_0^u H_s dB_s + \int_u^t H_s dB_s, \quad 0 \leq u \leq t.$$

$M_t := \int_0^t H_s dB_s$  is adapted and has continuous paths.

Past accumulated gains are known; future gains remain random.

This permits a running account of how reviews change the value of the consumer's option.

## Properties: No Predictable Gain from the Noise

### Proposition

If  $\mathbb{E} \int_0^T H_s^2 ds < \infty$ , then  $M_t = \int_0^t H_s dB_s$ ,  $t \leq T$ , is a square-integrable martingale. For  $u \leq t \leq T$ ,

$$\mathbb{E}[M_t - M_u \mid \mathcal{F}_u] = 0,$$

$$\mathbb{E}[(M_t - M_u)^2 \mid \mathcal{F}_u] = \mathbb{E} \left[ \int_u^t H_s^2 ds \mid \mathcal{F}_u \right].$$

$\mathbb{E}M_t = 0$ ;  $M_t^2 - \int_0^t H_s^2 ds$  is a martingale.

Review sensitivity may respond to all previous news without creating predictable gains from the next innovation.

The consumer's expected benefit must come from curvature, not from anticipating the sign of news.

## Properties: Variation and Covariation

For  $M = \int H dB$  and  $N = \int K dB$ ,

$$[M]_t = \int_0^t H_s^2 ds, \quad [M, N]_t = \int_0^t H_s K_s ds.$$

Consequently, for square-integrable integrands,

$$\mathbb{E}[M_t N_t] = \mathbb{E} \int_0^t H_s K_s ds.$$

## Properties: Variation and Covariation

For  $M = \int H dB$  and  $N = \int K dB$ ,

$$[M]_t = \int_0^t H_s^2 ds, \quad [M, N]_t = \int_0^t H_s K_s ds.$$

Consequently, for square-integrable integrands,

$$\mathbb{E}[M_t N_t] = \mathbb{E} \int_0^t H_s K_s ds.$$

Same review, same-sign sensitivities: positive covariation.

Independent Brownian news sources have zero cross-variation.

This is what will allow us to keep track of multiple sources of uncertainty in learning and control problems.

## Properties: Stopping the Flow of News

For a stopping time  $\tau$ ,

$$M_{t \wedge \tau} = \int_0^t \mathbf{1}_{\{s \leq \tau\}} H_s \, dB_s.$$

Stop accumulating innovations when the consumer buys or walks away.

## Properties: Stopping the Flow of News

For a stopping time  $\tau$ ,

$$M_{t \wedge \tau} = \int_0^t \mathbf{1}_{\{s \leq \tau\}} H_s dB_s.$$

Stop accumulating innovations when the consumer buys or walks away.

If  $\mathbb{E} \int_0^\tau H_s^2 ds < \infty$ , then

$$\mathbb{E} \int_0^\tau H_s dB_s = \mathbf{0}, \quad \mathbb{E} \left[ \left( \int_0^\tau H_s dB_s \right)^2 \right] = \mathbb{E} \int_0^\tau H_s^2 ds.$$

For the consumer,  $|\sigma \tilde{V}'| \leq \sigma$  and admissibility requires  $\mathbb{E} \tau < \infty$ .

Therefore the stochastic term will still have zero expectation at the decision time.

## What These Properties Do Not Say

Deterministic  $H$ :  $\int_0^t H_s \, dB_s$  is normal with variance  $\int_0^t H_s^2 \, ds$ .

Random adapted  $H$ : the integral need not be normal.

$$\int_0^t B_s \, dB_s = \frac{1}{2}(B_t^2 - t)$$

is bounded below by  $-t/2$  and is not normal.

## What These Properties Do Not Say

Deterministic  $H$ :  $\int_0^t H_s dB_s$  is normal with variance  $\int_0^t H_s^2 ds$ .

Random adapted  $H$ : the integral need not be normal.

$$\int_0^t B_s dB_s = \frac{1}{2}(B_t^2 - t)$$

is bounded below by  $-t/2$  and is not normal.

If only  $\int_0^T H_s^2 ds < \infty$  a.s., the integral is a **local martingale**.

There are stopping times  $\tau_n \uparrow \infty$  making each stopped process a martingale.

Localisation alone does not justify taking expectations at an unbounded stopping time:  
check integrability or uniform integrability.

# Itô's Formula for the Consumer's Valuation

## Theorem (Itô's Formula)

For  $X_t = x + \sigma B_t$  and  $\phi \in C^2$ ,

$$\phi(X_t) = \phi(x) + \int_0^t \sigma \phi'(X_s) dB_s + \int_0^t \frac{\sigma^2}{2} \phi''(X_s) ds.$$

First integral: unpredictable effect of favourable and unfavourable reviews.

Second integral: predictable gain from curvature.

# Itô's Formula for the Consumer's Valuation

## Theorem (Itô's Formula)

For  $X_t = x + \sigma B_t$  and  $\phi \in C^2$ ,

$$\phi(X_t) = \phi(x) + \int_0^t \sigma \phi'(X_s) dB_s + \int_0^t \frac{\sigma^2}{2} \phi''(X_s) ds.$$

First integral: unpredictable effect of favourable and unfavourable reviews.

Second integral: predictable gain from curvature.

Taking  $\phi(x) = x^2$  and  $X = B$  gives  $B_t^2 = 2 \int_0^t B_s dB_s + t$ .

Ordinary calculus loses the  $t$  term because it discards squared increments.

The formula also applies to  $C^1$ , piecewise  $C^2$  functions with locally bounded second derivative: no derivative jump, hence no local-time term.

## No Stopping Rule Can Do Better

Recall  $b = \sigma^2/(4c)$  and the payoff  $\tilde{V}$  of first exit from  $(-b, b)$ .

It is  $C^1$ , dominates  $v$ , and satisfies, away from the boundaries,

$$\frac{\sigma^2}{2}\tilde{V}'' - c = \begin{cases} 0, & |x| < b, \\ -c, & |x| > b. \end{cases}$$

## No Stopping Rule Can Do Better

Recall  $b = \sigma^2/(4c)$  and the payoff  $\tilde{V}$  of first exit from  $(-b, b)$ .

It is  $C^1$ , dominates  $v$ , and satisfies, away from the boundaries,

$$\frac{\sigma^2}{2}\tilde{V}'' - c = \begin{cases} 0, & |x| < b, \\ -c, & |x| > b. \end{cases}$$

Itô at any admissible  $\tau$ , with the stopping property above, gives

$$\begin{aligned} \mathbb{E}_x[v(X_\tau) - c\tau] &\leq \mathbb{E}_x[\tilde{V}(X_\tau) - c\tau] \\ &= \tilde{V}(x) + \mathbb{E}_x \int_0^\tau \left( \frac{\sigma^2}{2}\tilde{V}''(X_s) - c \right) ds \\ &\leq \tilde{V}(x). \end{aligned}$$

Thus  $V \leq \tilde{V}$ : the bound covers history-dependent rules too.

### Proposition

Let  $b = \sigma^2/(4c)$ . The consumer rejects at  $x \leq -b$ , searches at  $-b < x < b$ , and buys at  $x \geq b$ . The first exit  $\tau^*$  from  $(-b, b)$  is optimal and

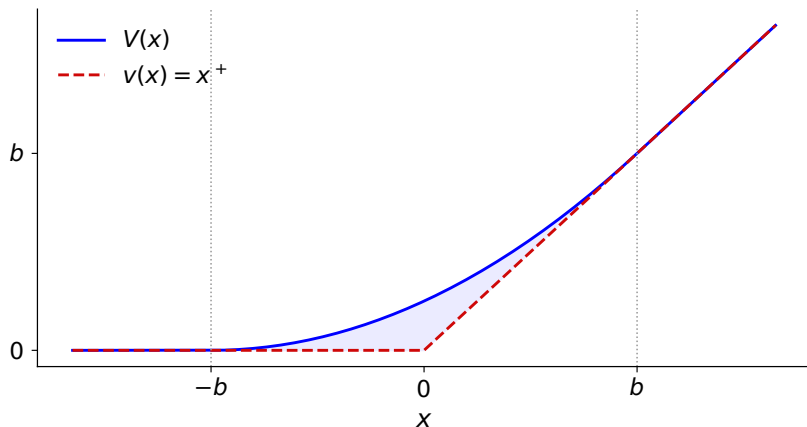
$$V(x) = \begin{cases} 0, & x \leq -b, \\ \frac{c}{\sigma^2}(x+b)^2, & -b < x < b, \\ x, & x \geq b. \end{cases}$$

### Proof

$\tau^*$  has finite expected duration. Before stopping, the drift of  $\tilde{V}(X_t) - ct$  is zero; at stopping,  $\tilde{V} = v$ . Both inequalities in the preceding bound are equalities.

Positive expected net utility is insufficient for purchase: waiting retains the option to reject after bad news.

# The Value of Another Review



$V - v$  is the value of keeping both decisions available.

Close to indifference, additional information can change the decision.

Far from indifference, that option is too small to justify the search cost.

## Pricing When Consumers Can Search

Allow the seller to choose price before search. Baseline valuation  $u$ , price  $p$ , marginal cost  $k$ .

With  $b = \sigma^2/(4c)$ , initial net utility is  $x = u - p$  and demand is

$$D(p) = \begin{cases} 1, & p \leq u - b, \\ (u - p + b)/(2b), & u - b < p < u + b, \\ 0, & p \geq u + b. \end{cases}$$

Seller maximises  $(p - k)D(p)$  over  $p \geq k$ .

## Pricing When Consumers Can Search

Allow the seller to choose price before search. Baseline valuation  $u$ , price  $p$ , marginal cost  $k$ .

With  $b = \sigma^2/(4c)$ , initial net utility is  $x = u - p$  and demand is

$$D(p) = \begin{cases} 1, & p \leq u - b, \\ (u - p + b)/(2b), & u - b < p < u + b, \\ 0, & p \geq u + b. \end{cases}$$

Seller maximises  $(p - k)D(p)$  over  $p \geq k$ .

If  $u + b \leq k$ : no strictly profitable sale.

Otherwise,

$$p^* = \begin{cases} (u + k + b)/2, & -b < u - k < 3b, \\ u - b, & u - k \geq 3b. \end{cases}$$

In the second case, price so that the consumer buys immediately.

## Lower Search Costs Can Hurt the Consumer

Hold the price fixed: lower  $c$  expands feasible gains from search.

But the seller adjusts its price. Write  $d = u - k > 0$ .

In the interior pricing region,  $b > d/3$ ,

$$p^* = k + \frac{d+b}{2}, \quad x^* = \frac{d-b}{2}, \quad V(x^*) = \frac{(d+b)^2}{16b}.$$

## Lower Search Costs Can Hurt the Consumer

Hold the price fixed: lower  $c$  expands feasible gains from search.

But the seller adjusts its price. Write  $d = u - k > 0$ .

In the interior pricing region,  $b > d/3$ ,

$$p^* = k + \frac{d+b}{2}, \quad x^* = \frac{d-b}{2}, \quad V(x^*) = \frac{(d+b)^2}{16b}.$$

Since  $\partial V(x^*)/\partial b = (1 - d^2/b^2)/16$ ,

$d/3 < b < d \implies \text{lower } c \implies \text{lower consumer surplus.}$

The price response can outweigh the direct benefit of cheaper information.

Individual and equilibrium comparative statics need not agree.

# Overview

1. Search for Product Information
2. Itô Calculus: The Value of Further Information
3. Dynamic Programming and Free Boundaries
4. Investment: The Value of Waiting
5. Entry and Exit: Hysteresis
6. Strategic Stopping: Races and Contests

## What Made the Argument Work?

In the search region, expected growth in continuation value exactly pays for another review:

$$\frac{\sigma^2}{2} V'' = c.$$

At the decision boundaries, continuation value meets the relevant scrap value tangentially.

## What Made the Argument Work?

In the search region, expected growth in continuation value exactly pays for another review:

$$\frac{\sigma^2}{2} V'' = c.$$

At the decision boundaries, continuation value meets the relevant scrap value tangentially.

Outside the search region, waiting incurs a cost without a compensating local benefit.

Can use these conditions to *find* the value, rather than first compute the payoff of a stopping rule.

Generalise to drift, state-dependent uncertainty, discounting, and flow payoffs.

## Definition

An **Itô process** has the form

$$X_t = X_0 + \int_0^t a_s \, ds + \int_0^t b_s \, dB_s, \quad \int_0^T (|a_s| + b_s^2) \, ds < \infty \text{ a.s.}$$

Here the coefficients are predictable. Write  $dX_t = a_t \, dt + b_t \, dB_t$ .

A time-homogeneous **diffusion** satisfies  $dX_t = \mu(X_t) \, dt + \sigma(X_t) \, dB_t$ .

Under standard existence and uniqueness assumptions (e.g. Lipschitz coefficients with linear growth), it is strong Markov.

## Definition

An **Itô process** has the form

$$X_t = X_0 + \int_0^t a_s ds + \int_0^t b_s dB_s, \quad \int_0^T (|a_s| + b_s^2) ds < \infty \text{ a.s.}$$

Here the coefficients are predictable. Write  $dX_t = a_t dt + b_t dB_t$ .

A time-homogeneous **diffusion** satisfies  $dX_t = \mu(X_t) dt + \sigma(X_t) dB_t$ .

Under standard existence and uniqueness assumptions (e.g. Lipschitz coefficients with linear growth), it is strong Markov.

Drift: expected change per unit time. Squared diffusion coefficient: local variance per unit time.

Consumer:  $\mu = 0$ , constant  $\sigma$ . Investment: both depend on current project value.

## Itô's Formula with Drift and Time

### Theorem

If  $dX_t = \mu(X_t) dt + \sigma(X_t) dB_t$  and  $\phi \in C^{1,2}$ ,

$$d\phi(t, X_t) = \left( \phi_t + \mu\phi_x + \frac{1}{2}\sigma^2\phi_{xx} \right) (t, X_t) dt \\ + \sigma(X_t)\phi_x(t, X_t) dB_t.$$

Three predictable effects: passage of time, drift in the state, and curvature under uncertainty.

## Itô's Formula with Drift and Time

### Theorem

If  $dX_t = \mu(X_t) dt + \sigma(X_t) dB_t$  and  $\phi \in C^{1,2}$ ,

$$d\phi(t, X_t) = \left( \phi_t + \mu\phi_x + \frac{1}{2}\sigma^2\phi_{xx} \right) (t, X_t) dt + \sigma(X_t)\phi_x(t, X_t) dB_t.$$

Three predictable effects: passage of time, drift in the state, and curvature under uncertainty.

For discounted value,  $\phi(t, x) = e^{-rt}W(x)$ ,

$$d(e^{-rt}W(X_t)) = e^{-rt} \left( \mu W' + \frac{1}{2}\sigma^2 W'' - rW \right) (X_t) dt + e^{-rt}\sigma(X_t)W'(X_t) dB_t.$$

Waiting must compensate for the opportunity cost  $rW$  as well as any direct flow cost.

# The Infinitesimal Generator

## Definition

The generator of a time-homogeneous Markov process is

$$\mathcal{L}\phi(x) := \lim_{h \downarrow 0} \frac{\mathbb{E}_x[\phi(X_h)] - \phi(x)}{h}$$

for functions in its domain. For a diffusion and suitable  $C^2$  functions,

$$\mathcal{L}\phi = \mu\phi' + \frac{1}{2}\sigma^2\phi''.$$

Expected rate of change of the object being valued.

$\mathcal{L}V - rV + f$ : expected net gain per unit time from delaying a decision, including flow payoff  $f$ .

Consumer:  $\mathcal{L}V - c = \sigma^2 V''/2 - c$ .

## Theorem

Under conditions making the stopped Itô integral mean zero and the terms integrable,

$$\mathbb{E}_x[e^{-r\tau}\phi(X_\tau)] = \phi(x) + \mathbb{E}_x \int_0^\tau e^{-rs}(\mathcal{L} - r)\phi(X_s) ds.$$

Accumulated expected gains equal the integral of their local rate.

For example, a bounded horizon and bounded  $\phi, \phi', \phi''$ , drift, and volatility suffice.

## Theorem

Under conditions making the stopped Itô integral mean zero and the terms integrable,

$$\mathbb{E}_x[e^{-r\tau}\phi(X_\tau)] = \phi(x) + \mathbb{E}_x \int_0^\tau e^{-rs}(\mathcal{L} - r)\phi(X_s) ds.$$

Accumulated expected gains equal the integral of their local rate.

For example, a bounded horizon and bounded  $\phi, \phi', \phi''$ , drift, and volatility suffice.

An exit probability solves  $\mathcal{L}p = 0$  with boundary values  $0, 1$ .

An expected exit time solves  $\mathcal{L}m = -1$  with zero boundary values.

This recovers the consumer's purchase probability and duration through differential equations.

# The General Stopping Problem

Strong Markov state  $X$ , flow payoff  $f$ , discount rate  $r \geq 0$ , scrap value  $v$ :

$$V(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x \left[ \int_0^\tau e^{-rs} f(X_s) ds + e^{-r\tau} v(X_\tau) \right].$$

Specify admissibility so the payoff is defined and finite; state the payoff convention if  $\tau = \infty$  is allowed.

# The General Stopping Problem

Strong Markov state  $X$ , flow payoff  $f$ , discount rate  $r \geq 0$ , scrap value  $v$ :

$$V(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x \left[ \int_0^\tau e^{-rs} f(X_s) ds + e^{-r\tau} v(X_\tau) \right].$$

Specify admissibility so the payoff is defined and finite; state the payoff convention if  $\tau = \infty$  is allowed.

**Stopping region:**  $D = \{x : V(x) = v(x)\}$ .

**Continuation region:**  $C = \{x : V(x) > v(x)\}$ .

Consumer:  $f = -c, r = 0, v = x^+, C = (-b, b)$ .

These are economic regions: does retaining the decision have positive value?

# The Dynamic Programming Principle

For a bounded stopping time  $\theta$ , under the usual regularity and integrability conditions,

$$V(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x \left[ \int_0^{\tau \wedge \theta} e^{-rs} f(X_s) ds + \mathbf{1}_{\{\tau \leq \theta\}} e^{-r\tau} V(X_\tau) + \mathbf{1}_{\{\tau > \theta\}} e^{-r\theta} V(X_\theta) \right].$$

Either stop before the next review date or arrive there with the option to act optimally afterwards.

# The Dynamic Programming Principle

For a bounded stopping time  $\theta$ , under the usual regularity and integrability conditions,

$$V(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x \left[ \int_0^{\tau \wedge \theta} e^{-rs} f(X_s) ds + \mathbf{1}_{\{\tau \leq \theta\}} e^{-r\tau} V(X_\tau) + \mathbf{1}_{\{\tau > \theta\}} e^{-r\theta} V(X_\theta) \right].$$

Either stop before the next review date or arrive there with the option to act optimally afterwards.

If waiting until  $\theta$  is optimal,

$$V(x) = \mathbb{E}_x \left[ \int_0^\theta e^{-rs} f(X_s) ds + e^{-r\theta} V(X_\theta) \right].$$

Locally inside  $C$ , take  $\theta = h \wedge \tau_U$  for a neighbourhood  $U$  whose closure lies in  $C$ .

The Markov state summarises what tomorrow's self needs to know.

# The Stopping Equation

Can stop immediately:  $V \geq v$ .

Can wait briefly and then act optimally:  $(\mathcal{L} - r)V + f \leq 0$ .

Where stopping is suboptimal, the dynamic programming equality gives  $(\mathcal{L} - r)V + f = 0$ .

# The Stopping Equation

Can stop immediately:  $V \geq v$ .

Can wait briefly and then act optimally:  $(\mathcal{L} - r)V + f \leq 0$ .

Where stopping is suboptimal, the dynamic programming equality gives  $(\mathcal{L} - r)V + f = 0$ .

$$\max\{v - V, (\mathcal{L} - r)V + f\} = 0.$$

The **HJB variational inequality** combines these restrictions.

Consumer:  $\max\{x^+ - V, \sigma^2 V''/2 - c\} = 0$ .

# The Stopping Equation

Can stop immediately:  $V \geq v$ .

Can wait briefly and then act optimally:  $(\mathcal{L} - r)V + f \leq 0$ .

Where stopping is suboptimal, the dynamic programming equality gives  $(\mathcal{L} - r)V + f = 0$ .

$$\max\{v - V, (\mathcal{L} - r)V + f\} = 0.$$

The **HJB variational inequality** combines these restrictions.

Consumer:  $\max\{x^+ - V, \sigma^2 V''/2 - c\} = 0$ .

Classical derivatives are used where they exist; boundary conditions and verification complete the argument.

Later: viscosity solutions when classical smoothness is unavailable.

## Value Matching and Smooth Fit

At a regular boundary  $b$  between stopping and waiting,

$$V(b) = v(b) \quad (\text{value matching}).$$

With a nondegenerate one-dimensional diffusion and differentiable scrap value at a regular optimal boundary, standard smooth-fit conditions give

$$V'(b) = v'(b) \quad (\text{smooth fit}).$$

## Value Matching and Smooth Fit

At a regular boundary  $b$  between stopping and waiting,

$$V(b) = v(b) \quad (\text{value matching}).$$

With a nondegenerate one-dimensional diffusion and differentiable scrap value at a regular optimal boundary, standard smooth-fit conditions give

$$V'(b) = v'(b) \quad (\text{smooth fit}).$$

At the margin, changing the state has the same effect on the best continuation plan and on immediate action.

Consumer:  $V'(-b) = 0$  and  $V'(b) = 1$ .

These conditions locate free boundaries; they are not a substitute for verifying optimality.

Degenerate processes, payoff kinks, or irregular boundaries require care.

## Recovering the Consumer's Thresholds

Within a connected search interval  $(a, b)$  containing  $0$ ,

$$\frac{\sigma^2}{2} \tilde{V}'' = c \implies \tilde{V}(x) = \frac{c}{\sigma^2} x^2 + Ax + D.$$

At rejection:  $\tilde{V}(a) = \tilde{V}'(a) = 0$ , so  $\tilde{V}(x) = c(x - a)^2 / \sigma^2$ .

## Recovering the Consumer's Thresholds

Within a connected search interval  $(a, b)$  containing  $0$ ,

$$\frac{\sigma^2}{2} \tilde{V}'' = c \implies \tilde{V}(x) = \frac{c}{\sigma^2} x^2 + Ax + D.$$

At rejection:  $\tilde{V}(a) = \tilde{V}'(a) = 0$ , so  $\tilde{V}(x) = c(x - a)^2 / \sigma^2$ .

At purchase:  $\tilde{V}(b) = b$  and  $\tilde{V}'(b) = 1$  imply

$$b - a = \frac{\sigma^2}{2c}, \quad b = \frac{\sigma^2}{4c}, \quad a = -b.$$

Same policy as before. We have already checked that it beats *all* feasible stopping rules.

Smooth fit balances the marginal value of further information at each decision boundary.

# Overview

1. Search for Product Information
2. Itô Calculus: The Value of Further Information
3. Dynamic Programming and Free Boundaries
4. Investment: The Value of Waiting
5. Entry and Exit: Hysteresis
6. Strategic Stopping: Races and Contests

# An Irreversible Investment

**McDonald & Siegel (1986 QJE); Dixit & Pindyck (1994, ch. 5)**

Pay  $I > 0$  once to acquire a project worth  $X_t$ .

Project value follows geometric Brownian motion:

$$dX_t = \mu X_t dt + \sigma X_t dB_t, \quad \sigma > 0, \quad r > \max\{\mu, 0\}.$$

# An Irreversible Investment

**McDonald & Siegel (1986 QJE); Dixit & Pindyck (1994, ch. 5)**

Pay  $I > 0$  once to acquire a project worth  $X_t$ .

Project value follows geometric Brownian motion:

$$dX_t = \mu X_t dt + \sigma X_t dB_t, \quad \sigma > 0, \quad r > \max\{\mu, 0\}.$$

May abandon for zero. Scrap value  $v(x) = (x - I)^+$  and

$$V(x) = \sup_{\tau} \mathbb{E}_x[e^{-r\tau} v(X_{\tau})].$$

Allow  $\tau = \infty$  with zero payoff. This is equivalent to the investment problem with payoff  $X_{\tau} - I$  and the option never to invest.

Positive net present value does not settle whether to invest *now*.

## Project Value and the Cost of Delay

Itô's formula gives

$$X_t = xZ_t, \quad Z_t = \exp \left( (\mu - \sigma^2/2)t + \sigma B_t \right).$$

Thus  $X_t > 0$  and  $\mathbb{E}_x X_t = xe^{\mu t}$ .

The generator is  $\mathcal{L}\phi = \mu x\phi' + \sigma^2 x^2 \phi''/2$ .

## Project Value and the Cost of Delay

Itô's formula gives

$$X_t = xZ_t, \quad Z_t = \exp \left( (\mu - \sigma^2/2)t + \sigma B_t \right).$$

Thus  $X_t > 0$  and  $\mathbb{E}_x X_t = xe^{\mu t}$ .

The generator is  $\mathcal{L}\phi = \mu x\phi' + \sigma^2 x^2 \phi''/2$ .

Since  $r > \mu$ ,  $e^{-rt}X_t$  is a nonnegative supermartingale.

Waiting sacrifices expected discounted project value but postpones the sunk cost and preserves the option to avoid bad outcomes.

The problem is stationary and Markovian: the decision depends on current  $x$ .

## Why the Stopping Region Must Be One-Sided

At  $x \leq l$ , immediate investment loses money or breaks even.

For every  $x > 0$ , waiting a fixed positive time gives a positive probability of  $X_t > l$ , hence  $V(x) > 0$ .

Therefore no stopping at  $0 < x \leq l$ : even abandonment destroys a valuable option.

## Why the Stopping Region Must Be One-Sided

At  $x \leq l$ , immediate investment loses money or breaks even.

For every  $x > 0$ , waiting a fixed positive time gives a positive probability of  $X_t > l$ , hence  $V(x) > 0$ .

Therefore no stopping at  $0 < x \leq l$ : even abandonment destroys a valuable option.

Couple initial states using the same  $Z$ , so  $X_t^x = xZ_t$ .

For each Brownian stopping time  $\tau$ , the expected payoff is increasing and convex in  $x$ .

Moreover, for  $y > x$ ,

$$0 \leq \mathbb{E}[e^{-r\tau}(yZ_\tau - l)^+] - \mathbb{E}[e^{-r\tau}(xZ_\tau - l)^+] \leq y - x,$$

$$\text{because } \mathbb{E}[e^{-r\tau}Z_\tau \mathbf{1}_{\{\tau < \infty\}}] \leq 1.$$

## A Single Kink Gives an Upper Stopping Region

### Proposition

$V$  is increasing, convex, and 1-Lipschitz. On  $[l, \infty)$ ,  $V(x) - (x - l)$  is nonnegative, convex, and nonincreasing. Thus if stopping is optimal at  $x$ , it is optimal at every  $y > x$ .

### Proof

Taking the supremum preserves convexity, monotonicity, and the common Lipschitz bound. For  $y > x \geq l$ ,

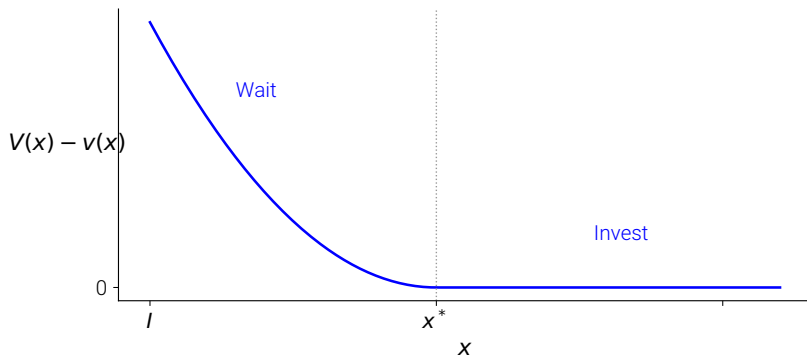
$$V(y) - (y - l) \leq V(x) - (x - l).$$

Once the nonnegative option premium reaches zero, it stays zero.

The one-kink scrap value and the process structure deliver an upper threshold; its shape is not a guess.

This argument allows an empty stopping region. The solution below establishes contact at a finite  $x^*$ .

## The Option Premium Vanishes Only Once



Above  $I$ , another pound raises scrap value by exactly one pound.

It raises the value of the investment option by at most one pound.

The gain from retaining flexibility therefore falls until exercise becomes optimal.

## Locating the Investment Threshold

Below an upper exercise boundary, the candidate must satisfy

$$\frac{1}{2}\sigma^2x^2\tilde{V}'' + \mu x\tilde{V}' - r\tilde{V} = 0.$$

Power solutions  $x^\beta$  have

$$\frac{1}{2}\sigma^2\beta(\beta - 1) + \mu\beta - r = 0, \quad \beta_- < 0 < 1 < \beta_+.$$

Since  $0 \leq V(x) \leq x$ , the negative-power term is excluded near zero.

## Locating the Investment Threshold

Below an upper exercise boundary, the candidate must satisfy

$$\frac{1}{2}\sigma^2x^2\tilde{V}'' + \mu x\tilde{V}' - r\tilde{V} = 0.$$

Power solutions  $x^\beta$  have

$$\frac{1}{2}\sigma^2\beta(\beta - 1) + \mu\beta - r = 0, \quad \beta_- < 0 < 1 < \beta_+.$$

Since  $0 \leq V(x) \leq x$ , the negative-power term is excluded near zero.

Write  $\beta = \beta_+$ . Value matching and smooth fit give

$$A(x^*)^\beta = x^* - l, \quad A\beta(x^*)^{\beta-1} = 1.$$

$$\text{Hence } x^* = \frac{\beta}{\beta - 1}l > l.$$

## Verifying the Investment Decision

Define

$$\tilde{V}(x) = \begin{cases} (x^* - I)(x/x^*)^\beta, & 0 < x < x^*, \\ x - I, & x \geq x^*. \end{cases}$$

It is  $C^1$ , nonnegative, and dominates  $v$ ; the convex power curve lies above its tangent  $x - I$ .

# Verifying the Investment Decision

Define

$$\tilde{V}(x) = \begin{cases} (x^* - l)(x/x^*)^\beta, & 0 < x < x^*, \\ x - l, & x \geq x^*. \end{cases}$$

It is  $C^1$ , nonnegative, and dominates  $v$ ; the convex power curve lies above its tangent  $x - l$ .

Below  $x^*$ ,  $(\mathcal{L} - r)\tilde{V} = 0$ .

Above it,

$$(\mathcal{L} - r)(x - l) = rl - (r - \mu)x \leq 0,$$

since  $(r - \mu)x^* - rl = \frac{1}{2}\sigma^2\beta l > 0$ .

The nonnegative discounted candidate is a supermartingale, so no feasible rule pays more than  $\tilde{V}$ .

## The Threshold Rule Attains the Bound

Let  $\tau^* = \inf\{t : X_t \geq x^*\}$ , with payoff zero if never hit.

For  $x < x^*$ , the discounted hitting-time transform is

$$\mathbb{E}_x[e^{-r\tau^*} \mathbf{1}_{\{\tau^* < \infty\}}] = \left(\frac{x}{x^*}\right)^\beta.$$

To see this, stop the local martingale  $e^{-rt}X_t^\beta$  at  $t \wedge \tau^*$ ; it is bounded by  $(x^*)^\beta$ .

On paths not yet stopped,  $e^{-rt}X_t^\beta \leq e^{-rt}(x^*)^\beta \rightarrow 0$ .

## The Threshold Rule Attains the Bound

Let  $\tau^* = \inf\{t : X_t \geq x^*\}$ , with payoff zero if never hit.

For  $x < x^*$ , the discounted hitting-time transform is

$$\mathbb{E}_x[e^{-r\tau^*} \mathbf{1}_{\{\tau^* < \infty\}}] = \left(\frac{x}{x^*}\right)^\beta.$$

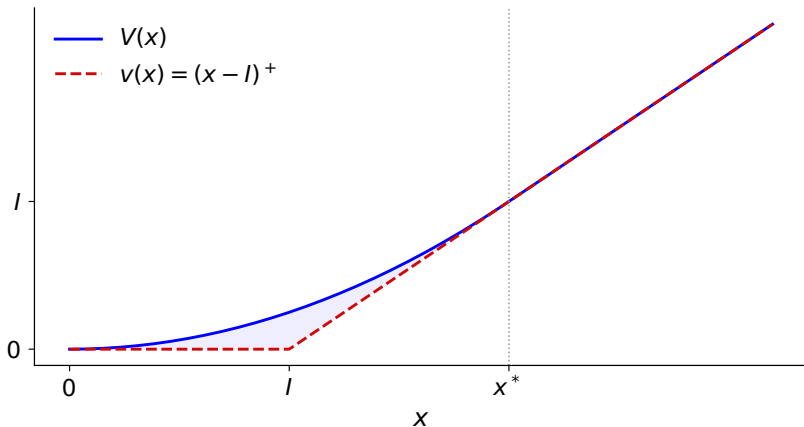
To see this, stop the local martingale  $e^{-rt}X_t^\beta$  at  $t \wedge \tau^*$ ; it is bounded by  $(x^*)^\beta$ .

On paths not yet stopped,  $e^{-rt}X_t^\beta \leq e^{-rt}(x^*)^\beta \rightarrow 0$ .

Thus the threshold rule earns exactly  $\tilde{V}(x)$ , and  $V = \tilde{V}$ .

Invest when  $x \geq x^*$ ; otherwise keep the option alive.

## Positive NPV Is Not Enough



At  $I < x < x^*$ , investment is profitable but waiting is better.

Exercise sacrifices  $V(x) - v(x)$ , the value of responding to future news.

The trigger exceeds cost by  $x^* - I = I/(\beta - 1)$ .

## Greater Uncertainty Raises the Investment Trigger

Let  $q(\beta) = \frac{1}{2}\sigma^2\beta(\beta - 1) + \mu\beta - r$ .

At its positive root,  $q_\beta > 0$ , so

$$\frac{\partial \beta}{\partial \sigma^2} = -\frac{\beta(\beta - 1)}{2q_\beta} < 0, \quad \frac{\partial x^*}{\partial \sigma^2} = -\frac{1}{(\beta - 1)^2} \frac{\partial \beta}{\partial \sigma^2} > 0.$$

## Greater Uncertainty Raises the Investment Trigger

Let  $q(\beta) = \frac{1}{2}\sigma^2\beta(\beta - 1) + \mu\beta - r$ .

At its positive root,  $q_\beta > 0$ , so

$$\frac{\partial \beta}{\partial \sigma^2} = -\frac{\beta(\beta - 1)}{2q_\beta} < 0, \quad \frac{\partial x^*}{\partial \sigma^2} = -\frac{1}{(\beta - 1)^2} \frac{\partial \beta}{\partial \sigma^2} > 0.$$

More uncertainty increases the value of retaining flexibility under irreversible investment.

The consumer waits for stronger evidence; the investor waits for a higher project value.

A higher trigger need not mean a longer expected delay under every process: volatility also changes the speed of reaching it.

# Overview

1. Search for Product Information
2. Itô Calculus: The Value of Further Information
3. Dynamic Programming and Free Boundaries
4. Investment: The Value of Waiting
5. Entry and Exit: Hysteresis
6. Strategic Stopping: Races and Contests

# Entry and Exit Decisions

## Setup (Dixit 1989 JPE)

Output price:  $dP_t = \mu P_t dt + \sigma P_t dB_t$ ,  $r > \max\{\mu, 0\}$ .

Active firm earns flow profit  $P_t - w$ ,  $w > 0$ .

Idle firm earns 0.

Entry costs  $I > 0$ ; exit costs  $E \geq 0$ ; re-entry is possible.

# Entry and Exit Decisions

## Setup (Dixit 1989 JPE)

Output price:  $dP_t = \mu P_t dt + \sigma P_t dB_t$ ,  $r > \max\{\mu, 0\}$ .

Active firm earns flow profit  $P_t - w$ ,  $w > 0$ .

Idle firm earns 0.

Entry costs  $I > 0$ ; exit costs  $E \geq 0$ ; re-entry is possible.

$V_0(p)$ : value when idle;  $V_1(p)$ : value when active.

Switching changes the regime, not the price process.

Entry and exit are linked stopping problems.

## Two Linked Stopping Equations

Idle: wait or take scrap value  $v_0 = V_1 - I$  by entering.

$$\max\{V_1 - I - V_0, (\mathcal{L} - r)V_0\} = 0.$$

Active: operate or take scrap value  $v_1 = V_0 - E$  by exiting.

$$\max\{V_0 - E - V_1, (\mathcal{L} - r)V_1 + p - w\} = 0.$$

## Two Linked Stopping Equations

Idle: wait or take scrap value  $v_0 = V_1 - I$  by entering.

$$\max\{V_1 - I - V_0, (\mathcal{L} - r)V_0\} = 0.$$

Active: operate or take scrap value  $v_1 = V_0 - E$  by exiting.

$$\max\{V_0 - E - V_1, (\mathcal{L} - r)V_1 + p - w\} = 0.$$

With linear operating profits and sunk switching costs, consider entry at  $p_H$  and exit at  $p_L < p_H$ ; verify both regimes below.

For  $p_L < p < p_H$ , action depends on whether the firm is already active.

The state is  $(p, \text{regime})$ , not just  $p$ .

## Solving for the Entry and Exit Triggers

Let  $\beta_- < 0 < 1 < \beta_+$  solve the same characteristic equation as before.

In each regime's continuation region,

$$V_0(p) = Ap^{\beta_+}, \quad V_1(p) = \frac{p}{r - \mu} - \frac{w}{r} + Bp^{\beta_-}.$$

The last term is the option to exit; it vanishes as  $p \rightarrow \infty$ .

## Solving for the Entry and Exit Triggers

Let  $\beta_- < 0 < 1 < \beta_+$  solve the same characteristic equation as before.

In each regime's continuation region,

$$V_0(p) = Ap^{\beta_+}, \quad V_1(p) = \frac{p}{r - \mu} - \frac{w}{r} + Bp^{\beta_-}.$$

The last term is the option to exit; it vanishes as  $p \rightarrow \infty$ .

Value matching and smooth fit give four equations:

$$\begin{aligned} V_0(p_H) &= V_1(p_H) - I, & V'_0(p_H) &= V'_1(p_H), \\ V_1(p_L) &= V_0(p_L) - E, & V'_1(p_L) &= V'_0(p_L). \end{aligned}$$

Solve for  $A, B, p_L, p_H$ ; check the regime inequalities.

If exit is too costly, an interior exit trigger need not exist.

# Hysteresis

Consider  $w > rE$ , so exit can be worthwhile at low prices.

At the entry boundary, waiting must be unattractive to the idle firm:

$$(\mathcal{L} - r)(V_1 - l) = -(p - w) + rl \leq 0 \quad \implies \quad p_H \geq w + rl.$$

# Hysteresis

Consider  $w > rE$ , so exit can be worthwhile at low prices.

At the entry boundary, waiting must be unattractive to the idle firm:

$$(\mathcal{L} - r)(V_1 - I) = -(p - w) + rI \leq 0 \implies p_H \geq w + rI.$$

At the exit boundary, continued operation must be unattractive:

$$(\mathcal{L} - r)(V_0 - E) + p - w = rE + p - w \leq 0 \implies p_L \leq w - rE.$$

With nondegenerate uncertainty and interior triggers, the standard solution has strict inequalities.

# Hysteresis

Consider  $w > rE$ , so exit can be worthwhile at low prices.

At the entry boundary, waiting must be unattractive to the idle firm:

$$(\mathcal{L} - r)(V_1 - I) = -(p - w) + rI \leq 0 \implies p_H \geq w + rI.$$

At the exit boundary, continued operation must be unattractive:

$$(\mathcal{L} - r)(V_0 - E) + p - w = rE + p - w \leq 0 \implies p_L \leq w - rE.$$

With nondegenerate uncertainty and interior triggers, the standard solution has strict inequalities.

**Hysteresis:** the same current price can sustain an active or an idle firm, depending on past entry and exit.

## Entry and Exit: A Numerical Example

$$r = .05, \mu = 0, \sigma = .20, w = 1, l = 2, E = 1.$$

Solving the four boundary equations gives

$$p_L \simeq 0.651, \quad p_H \simeq 1.675.$$

Compare with  $w - rE = .95$  and  $w + rl = 1.10$ .

|                  | $p < p_L$ | $p_L < p < p_H$ | $p > p_H$ |
|------------------|-----------|-----------------|-----------|
| Currently idle   | Wait      | Wait            | Enter     |
| Currently active | Exit      | Operate         | Operate   |

Sunk costs and uncertainty create a region of inaction.

A temporary price shock can have a persistent effect on activity.

# Overview

1. Search for Product Information
2. Itô Calculus: The Value of Further Information
3. Dynamic Programming and Free Boundaries
4. Investment: The Value of Waiting
5. Entry and Exit: Hysteresis
6. Strategic Stopping: Races and Contests

# Strategic Delay in R&D

## **Weeds (2002 REStud)**

Two firms choose when to initiate irreversible research.

Patent value follows geometric Brownian motion.

Active research succeeds at a Poisson arrival rate.

The first discovery wins the patent.

# Strategic Delay in R&D

## Weeds (2002 REStud)

Two firms choose when to initiate irreversible research.

Patent value follows geometric Brownian motion.

Active research succeeds at a Poisson arrival rate.

The first discovery wins the patent.

Waiting preserves flexibility; investing can secure a lead.

A rival's entry reduces the value of an active project.

Competition can induce preemption, but also sustain joint delay.

The monopoly stopping threshold alone does not determine equilibrium.

# Leader and Follower Values

Compute payoffs backwards:

- Value when both firms conduct research.

- Follower's option to initiate research.

- Leader's value, anticipating follower entry.

## Leader and Follower Values

Compute payoffs backwards:

- Value when both firms conduct research.

- Follower's option to initiate research.

- Leader's value, anticipating follower entry.

Compare net leader and follower values,  $V_L(x)$  and  $V_F(x)$ .

Where  $V_L(x) > V_F(x)$ , each firm would prefer to move first.

Whether delaying together is sustainable also requires checking deviations.

## Leader and Follower Values

Compute payoffs backwards:

- Value when both firms conduct research.

- Follower's option to initiate research.

- Leader's value, anticipating follower entry.

Compare net leader and follower values,  $V_L(x)$  and  $V_F(x)$ .

Where  $V_L(x) > V_F(x)$ , each firm would prefer to move first.

Whether delaying together is sustainable also requires checking deviations.

Smooth fit applies at an optimally chosen follower boundary.

At the rival's boundary, the leader's value must match; the leader does not choose that boundary, so do not impose its smooth-fit condition there.

## Research Success as a Random Arrival

Symmetric race: entry cost  $I$ , research flow cost  $c_R$ , success rate  $\lambda$  per active firm, independent of patent value  $X$ .

In a short interval  $h$ , one active firm's success probability is  $\lambda h + o(h)$ .

With two active firms, the race ends at total rate  $2\lambda$ .

## Research Success as a Random Arrival

Symmetric race: entry cost  $I$ , research flow cost  $c_R$ , success rate  $\lambda$  per active firm, independent of patent value  $X$ .

In a short interval  $h$ , one active firm's success probability is  $\lambda h + o(h)$ .

With two active firms, the race ends at total rate  $2\lambda$ .

A firm's value after both have entered is

$$\begin{aligned} A_2(X) &= \mathbb{E}_X \int_0^\infty e^{-(r+2\lambda)t} (\lambda X_t - c_R) dt \\ &= \frac{\lambda X}{r + 2\lambda - \mu} - \frac{c_R}{r + 2\lambda}. \end{aligned}$$

Assume  $r > \max\{\mu, 0\}$ . The rival's possible success discounts future gains.

## The Follower's Stopping Problem

The follower can take scrap value  $v_F = A_2 - I$ . While it waits, leader success ends the opportunity at rate  $\lambda$ .

Therefore, before the follower's entry threshold  $x_F$ ,

$$(\mathcal{L} - r - \lambda)V_F = 0.$$

Write  $V_F(x) = Kx^\gamma$ , where  $\gamma > 1$  solves

$$\frac{1}{2}\sigma^2\gamma(\gamma - 1) + \mu\gamma - (r + \lambda) = 0.$$

## The Follower's Stopping Problem

The follower can take scrap value  $v_F = A_2 - I$ . While it waits, leader success ends the opportunity at rate  $\lambda$ .

Therefore, before the follower's entry threshold  $x_F$ ,

$$(\mathcal{L} - r - \lambda)V_F = 0.$$

Write  $V_F(x) = Kx^\gamma$ , where  $\gamma > 1$  solves

$$\frac{1}{2}\sigma^2\gamma(\gamma - 1) + \mu\gamma - (r + \lambda) = 0.$$

Set  $V_F(x_F) = A_2(x_F) - I$  and  $V'_F(x_F) = A'_2(x_F)$ :

$$x_F = \frac{\gamma}{\gamma - 1} \left( I + \frac{C_R}{r + 2\lambda} \right) \frac{r + 2\lambda - \mu}{\lambda}.$$

The stopping method survives; the rival changes both the payoff and the effective discount rate.

## The Leader Anticipates Entry

Let  $A_1(x)$  be the active leader's value before follower entry.

Until  $X$  reaches  $x_F$ ,

$$(\mathcal{L} - r - \lambda)A_1 + \lambda x - c_R = 0.$$

A solution satisfying the natural condition near zero is

$$A_1(x) = \frac{\lambda x}{r + \lambda - \mu} - \frac{c_R}{r + \lambda} + Cx^\gamma.$$

## The Leader Anticipates Entry

Let  $A_1(x)$  be the active leader's value before follower entry.

Until  $X$  reaches  $x_F$ ,

$$(\mathcal{L} - r - \lambda)A_1 + \lambda x - c_R = 0.$$

A solution satisfying the natural condition near zero is

$$A_1(x) = \frac{\lambda x}{r + \lambda - \mu} - \frac{c_R}{r + \lambda} + Cx^\gamma.$$

Determine  $C$  from  $A_1(x_F) = A_2(x_F)$ .

Net leader value before initiating research is  $V_L(x) = A_1(x) - I$ .

These values supply the timing game; choosing equilibrium initiation requires the leader–follower incentive comparisons.

# Gambling in Contests

## Setup (Seel & Strack 2013 JET)

$n \geq 2$  players privately observe independent processes  $X_t^i = x_0 + \mu t + \sigma B_t^i$ ,  $x_0 > 0$ .

Choose a possibly randomised stopping time  $\tau_i$ , with  $\mathbb{E}\tau_i < \infty$ .

Must stop by bankruptcy:  $\tau_i \leq \inf\{t : X_t^i = 0\}$ .

Highest stopped value wins a unit prize; ties split it.

# Gambling in Contests

## Setup (Seel & Strack 2013 JET)

$n \geq 2$  players privately observe independent processes  $X_t^i = x_0 + \mu t + \sigma B_t^i$ ,  $x_0 > 0$ .

Choose a possibly randomised stopping time  $\tau_i$ , with  $\mathbb{E}\tau_i < \infty$ .

Must stop by bankruptcy:  $\tau_i \leq \inf\{t : X_t^i = 0\}$ .

Highest stopped value wins a unit prize; ties split it.

Focus on  $\mu \leq 0$ .

Given atomless rival cdf  $G$ , the scrap value is  $v(x; G) = G(x)^{n-1}$ .

Rank incentives change the stopping payoff.

## A Driftless Contest

Let  $\mu = 0$ . Suppose symmetric equilibrium has upper support  $\bar{x}$ .

Indifference on the support requires a harmonic winning probability:

$$\frac{1}{2}\sigma^2 v'' = 0, \quad v(0) = 0, \quad v(\bar{x}) = 1.$$

Thus  $v(x) = x/\bar{x}$  on  $[0, \bar{x}]$ .

## A Driftless Contest

Let  $\mu = 0$ . Suppose symmetric equilibrium has upper support  $\bar{x}$ .

Indifference on the support requires a harmonic winning probability:

$$\frac{1}{2}\sigma^2 v'' = 0, \quad v(0) = 0, \quad v(\bar{x}) = 1.$$

Thus  $v(x) = x/\bar{x}$  on  $[0, \bar{x}]$ .

Symmetry:  $v(x_0) = 1/n$ , hence  $\bar{x} = nx_0$ .

$$G(x) = \left( \frac{x}{nx_0} \right)^{1/(n-1)}, \quad 0 \leq x \leq nx_0.$$

A candidate distribution is not yet a stopping strategy.

# Constructing an Equilibrium: Two Players

## Proposition ( $n = 2, \mu = 0$ )

Independently draw  $A_i \sim U(0, x_0)$  at time zero and stop on first exit from  $(x_0 - A_i, x_0 + A_i)$ . These strategies form an equilibrium.

## Proof

Conditional on  $A_i = a$ , each endpoint is hit with probability  $1/2$ . Mixing over  $a$  makes  $X_{\tau_i}^i$  uniform on  $[0, 2x_0]$ .

Against this distribution, the winning payoff is  $v(x) = \min\{x/(2x_0), 1\}$  for  $x \geq 0$ . It is concave, so  $v(X_{t \wedge \tau_0})$  is a bounded supermartingale.

Every deviation pays at most  $v(x_0) = 1/2$ ; the proposed rule attains  $1/2$ .

## Negative Drift and Risk Taking

For  $\mu < 0$ , define the scale function

$$s(x) = \frac{e^{-2\mu x/\sigma^2} - 1}{-2\mu/\sigma^2}, \quad \mathcal{L}s = 0.$$

Indifference gives  $v(x) = s(x)/s(\bar{x})$  and  $s(\bar{x}) = ns(x_0)$ .

Hence the candidate cdf on  $[0, \bar{x}]$  is

$$G(x) = \left( \frac{s(x)}{ns(x_0)} \right)^{1/(n-1)}.$$

## Negative Drift and Risk Taking

For  $\mu < 0$ , define the scale function

$$s(x) = \frac{e^{-2\mu x/\sigma^2} - 1}{-2\mu/\sigma^2}, \quad \mathcal{L}s = 0.$$

Indifference gives  $v(x) = s(x)/s(\bar{x})$  and  $s(\bar{x}) = ns(x_0)$ .

Hence the candidate cdf on  $[0, \bar{x}]$  is

$$G(x) = \left( \frac{s(x)}{ns(x_0)} \right)^{1/(n-1)}.$$

For  $n = 2$ , draw  $A \sim U(0, 1)$  and stop when  $s(X_t)/s(x_0)$  exits  $(1 - A, 1 + A)$ .

Optional sampling gives equal endpoint probabilities; the construction induces  $G$ . The payoff  $\min\{s(x)/(2s(x_0)), 1\}$  is a concave function of the stopped scale martingale, verifying the best response.

The winning payoff is convex on the interior support: rank incentives reward dispersion despite negative drift.

## Private Incentives and Expected Losses

For  $n = 2$  and  $\mu < 0$ , the constructed rule has bounded state support and finite expected duration.

Wald's identity gives

$$\mathbb{E}[X_\tau] = x_0 + \mu \mathbb{E}[\tau] < x_0.$$

## Private Incentives and Expected Losses

For  $n = 2$  and  $\mu < 0$ , the constructed rule has bounded state support and finite expected duration.

Wald's identity gives

$$\mathbb{E}[X_\tau] = x_0 + \mu \mathbb{E}[\tau] < x_0.$$

Maximising expected stopped value would favour stopping immediately.

Maximising the probability of winning induces continued gambling.

Zero expected drift removes expected losses, but not risk taking.

## Private Incentives and Expected Losses

For  $n = 2$  and  $\mu < 0$ , the constructed rule has bounded state support and finite expected duration.

Wald's identity gives

$$\mathbb{E}[X_\tau] = x_0 + \mu \mathbb{E}[\tau] < x_0.$$

Maximising expected stopped value would favour stopping immediately.

Maximising the probability of winning induces continued gambling.

Zero expected drift removes expected losses, but not risk taking.

The same tools now verify a best response to an endogenous payoff.

# Stopping Problems: A Working Method

1. Specify the state, information, feasible stopping rules, and payoffs.
2. Write the generator and the stopping equation.
3. Use payoff and process structure to derive the region; solve inside it.
4. Use value matching and, when justified, smooth fit.
5. Verify the payoff bound, drift inequalities, and limiting conditions.
6. Recover behaviour: exit probabilities, duration, comparative statics.

# Stopping Problems: A Working Method

1. Specify the state, information, feasible stopping rules, and payoffs.
2. Write the generator and the stopping equation.
3. Use payoff and process structure to derive the region; solve inside it.
4. Use value matching and, when justified, smooth fit.
5. Verify the payoff bound, drift inequalities, and limiting conditions.
6. Recover behaviour: exit probabilities, duration, comparative statics.

**Next:** when the drift is unknown, what state should the DM use?

Learning: derive the posterior process rather than assume its law.

## References: Economic Applications

- Branco, Sun, & Villas-Boas (2012 MnSc), *Optimal Search for Product Information*.
- McDonald & Siegel (1986 QJE), *The Value of Waiting to Invest*.
- Dixit (1989 JPE), *Entry and Exit Decisions under Uncertainty*.
- Dixit & Pindyck (1994), *Investment under Uncertainty*, ch. 5.
- Weeds (2002 REStud), *Strategic Delay in a Real Options Model of R&D Competition*.
- Seel & Strack (2013 JET), *Gambling in Contests*.

## References: Mathematical Background

Øksendal (2003), *Stochastic Differential Equations: An Introduction with Applications*, chs. 7 and 10.

Diffusions, generators, Dynkin's formula, and optimal stopping.

Pham (2009), *Continuous-time Stochastic Control and Optimization with Financial Applications*, chs. 1, 3, and 5.

Stochastic calculus, dynamic programming, and stopping.

Karatzas & Shreve (1998 printing), *Brownian Motion and Stochastic Calculus*, 2nd ed.

Further foundations and precise versions of the probabilistic results.

## Further Results and Proofs

Versions, stopping-time information, and strong Markov property.

Reflection, time scaling, and drifted exit probabilities.

General verification and the Snell envelope.

Finite horizons and failure of smooth fit.

# Modifications and Indistinguishability

$X, Y$  are **modifications** if  $\mathbb{P}(X_t = Y_t) = 1$  for each fixed  $t$ .

They are **indistinguishable** if  $\mathbb{P}(X_t = Y_t \ \forall t \geq 0) = 1$ .

# Modifications and Indistinguishability

$X, Y$  are **modifications** if  $\mathbb{P}(X_t = Y_t) = 1$  for each fixed  $t$ .

They are **indistinguishable** if  $\mathbb{P}(X_t = Y_t \ \forall t \geq 0) = 1$ .

Order of quantifiers matters: an uncountable union of null events need not be null.

Example:  $U \sim U(0, 1)$ ,  $X_t = \mathbf{1}_{\{t=U\}}$  and  $Y_t = 0$ .

At every fixed  $t$ ,  $X_t = Y_t$  a.s.; yet every path differs somewhere.

# Modifications and Indistinguishability

$X, Y$  are **modifications** if  $\mathbb{P}(X_t = Y_t) = 1$  for each fixed  $t$ .

They are **indistinguishable** if  $\mathbb{P}(X_t = Y_t \forall t \geq 0) = 1$ .

Order of quantifiers matters: an uncountable union of null events need not be null.

Example:  $U \sim U(0, 1)$ ,  $X_t = \mathbf{1}_{\{t=U\}}$  and  $Y_t = 0$ .

At every fixed  $t$ ,  $X_t = Y_t$  a.s.; yet every path differs somewhere.

Continuous modifications are indistinguishable.

Proof: equality simultaneously at rational times, then continuity.

### Definition

$$\mathcal{F}_\tau := \{A \in \mathcal{F} : A \cap \{\tau \leq t\} \in \mathcal{F}_t \forall t \geq 0\}.$$

Describes information available when stopping occurs.

For adapted continuous  $X$ ,  $X_\tau$  is  $\mathcal{F}_\tau$ -measurable on  $\{\tau < \infty\}$ .

# Information at a Stopping Time

## Definition

$$\mathcal{F}_\tau := \{A \in \mathcal{F} : A \cap \{\tau \leq t\} \in \mathcal{F}_t \forall t \geq 0\}.$$

Describes information available when stopping occurs.

For adapted continuous  $X$ ,  $X_\tau$  is  $\mathcal{F}_\tau$ -measurable on  $\{\tau < \infty\}$ .

If  $S, T$  are stopping times, so are  $S \wedge T$ ,  $S \vee T$ , and  $S + T$  (under the usual conditions).

For continuous adapted  $X$ , first hitting a closed set is a stopping time.

Thus the threshold rules used in the lecture are feasible.

## Restarting at a Random Time

### Theorem (Strong Markov Property of Brownian Motion)

For an a.s. finite stopping time  $\tau$ , the process

$$(B_{\tau+t} - B_{\tau})_{t \geq 0}$$

is a Brownian motion independent of  $\mathcal{F}_{\tau}$ .

Brownian motion restarts after hitting a boundary.

It does not restart after an arbitrary random time, such as a future maximum.

## Restarting at a Random Time

### Theorem (Strong Markov Property of Brownian Motion)

For an a.s. finite stopping time  $\tau$ , the process

$$(B_{\tau+t} - B_{\tau})_{t \geq 0}$$

is a Brownian motion independent of  $\mathcal{F}_{\tau}$ .

Brownian motion restarts after hitting a boundary.

It does not restart after an arbitrary random time, such as a future maximum.

Supports dynamic programming after state-dependent decisions.

For a strong Markov diffusion, the new law depends on the state  $X_{\tau}$ .

## Reflection and One-Sided Hitting Times

Let  $\tau_a = \inf\{t : B_t = a\}$  for  $a > 0$ .

Reflect a path after its first hit of  $a$ : symmetry and strong Markov preserve its probability law.

$$\mathbb{P}(\tau_a \leq t) = \mathbb{P}\left(\max_{s \leq t} B_s \geq a\right) = 2\mathbb{P}(B_t \geq a) = 2\left[1 - \Phi\left(\frac{a}{\sqrt{t}}\right)\right].$$

## Reflection and One-Sided Hitting Times

Let  $\tau_a = \inf\{t : B_t = a\}$  for  $a > 0$ .

Reflect a path after its first hit of  $a$ : symmetry and strong Markov preserve its probability law.

$$\mathbb{P}(\tau_a \leq t) = \mathbb{P}\left(\max_{s \leq t} B_s \geq a\right) = 2\mathbb{P}(B_t \geq a) = 2\left[1 - \Phi\left(\frac{a}{\sqrt{t}}\right)\right].$$

Differentiate to obtain the density

$$f_{\tau_a}(t) = \frac{a}{\sqrt{2\pi t^3}} e^{-a^2/(2t)}, \quad t > 0.$$

$\tau_a < \infty$  a.s., but  $\mathbb{E}\tau_a = \infty$ .

A stopping rule can eventually succeed and still have an infinite expected waiting time.

## Time Scaling

For  $X_t = x + \sigma B_t$  and fixed boundaries  $a < x < b$ ,

$$\tau_{a,b}^\sigma \stackrel{d}{=} \frac{1}{\sigma^2} \tau_{a,b}^1.$$

Multiplying volatility speeds up the same path distribution.

## Time Scaling

For  $X_t = x + \sigma B_t$  and fixed boundaries  $a < x < b$ ,

$$\tau_{a,b}^\sigma \stackrel{d}{=} \frac{1}{\sigma^2} \tau_{a,b}^1.$$

Multiplying volatility speeds up the same path distribution.

For the consumer problem, time change  $s = \sigma^2 t$  gives

$$\sup_{\tau} \mathbb{E}[(x + \sigma B_\tau)^+ - c\tau] = \sup_{\rho} \mathbb{E}[(x + B_\rho)^+ - (c/\sigma^2)\rho].$$

Policies depend on  $c/\sigma^2$  in Brownian time.

Calendar duration also divides by  $\sigma^2$ ; remember both effects.

## Exit Probabilities with Drift

$X_t = x + \mu t + \sigma B_t$ ; first exit  $\tau$  from  $(a, b)$ .

For  $\mu \neq 0$ , solve  $\mu p' + \frac{1}{2}\sigma^2 p'' = 0$  with  $p(a) = 0, p(b) = 1$ :

$$p(x) = \frac{1 - e^{-2\mu(x-a)/\sigma^2}}{1 - e^{-2\mu(b-a)/\sigma^2}}.$$

## Exit Probabilities with Drift

$X_t = x + \mu t + \sigma B_t$ ; first exit  $\tau$  from  $(a, b)$ .

For  $\mu \neq 0$ , solve  $\mu p' + \frac{1}{2}\sigma^2 p'' = 0$  with  $p(a) = 0, p(b) = 1$ :

$$p(x) = \frac{1 - e^{-2\mu(x-a)/\sigma^2}}{1 - e^{-2\mu(b-a)/\sigma^2}}.$$

Wald's identity, justified by  $\mathbb{E}\tau < \infty$ , gives

$$\mathbb{E}_x \tau = \frac{a + (b - a)p(x) - x}{\mu}.$$

As  $\mu \rightarrow 0$ , recover  $(x - a)/(b - a)$  and  $(x - a)(b - x)/\sigma^2$ .

The scale function converts the drifted exit problem into a linear one.

### Theorem (A Sufficient Verification Result)

Let  $\tilde{V} \geq v$  and let

$$Z_t = e^{-rt}\tilde{V}(X_t) + \int_0^t e^{-rs}f(X_s) ds$$

be a supermartingale. Assume optional sampling and passage to the terminal payoff are valid for every admissible rule. Then  $\tilde{V} \geq V$ .

If a feasible  $\tau^*$  satisfies  $\tilde{V}(X_{\tau^*}) = v(X_{\tau^*})$  and  $Z_{t \wedge \tau^*}$  is a uniformly integrable martingale, then  $\tilde{V} = V$  and  $\tau^*$  is optimal.

If  $\tau^* = \infty$  is possible, require the corresponding terminal limit and transversality condition.

Itô and  $(\mathcal{L} - r)\tilde{V} + f \leq 0$  establish the supermartingale bound under suitable integrability conditions.

# The Snell Envelope

Finite horizon  $T$ ; continuous adapted reward  $Y$  with  $\mathbb{E} \sup_{t \leq T} |Y_t| < \infty$ .

Define

$$S_t = \operatorname{ess\,sup}_{\tau \in \mathbb{T}, t \leq \tau \leq T} \mathbb{E}[Y_\tau \mid \mathcal{F}_t].$$

$S$  is the smallest supermartingale dominating  $Y$ .

# The Snell Envelope

Finite horizon  $T$ ; continuous adapted reward  $Y$  with  $\mathbb{E} \sup_{t \leq T} |Y_t| < \infty$ .

Define

$$S_t = \operatorname{ess\,sup}_{\tau \in \mathbb{T}, t \leq \tau \leq T} \mathbb{E}[Y_\tau \mid \mathcal{F}_t].$$

$S$  is the smallest supermartingale dominating  $Y$ .

$\tau^* = \inf\{t : S_t = Y_t\}$  is optimal;  $S_{t \wedge \tau^*}$  is a martingale.

For a Markov stopping problem,

$$Y_t = \int_0^t e^{-rs} f(X_s) ds + e^{-rt} v(X_t).$$

The free-boundary method is a concrete way to construct this payoff bound.

## Finite Horizons

With a deadline  $T$ , time is part of the state.

$$V(t, x) = \sup_{t \leq \tau \leq T} \mathbb{E}_{t,x} \left[ \int_t^\tau e^{-r(s-t)} f(s, X_s) ds + e^{-r(\tau-t)} v(\tau, X_\tau) \right].$$

The stopping equation becomes

$$\max\{v - V, V_t + \mathcal{L}V - rV + f\} = 0, \quad V(T, x) = v(T, x).$$

## Finite Horizons

With a deadline  $T$ , time is part of the state.

$$V(t, x) = \sup_{t \leq \tau \leq T} \mathbb{E}_{t,x} \left[ \int_t^\tau e^{-r(s-t)} f(s, X_s) ds + e^{-r(\tau-t)} v(\tau, X_\tau) \right].$$

The stopping equation becomes

$$\max\{v - V, V_t + \mathcal{L}V - rV + f\} = 0, \quad V(T, x) = v(T, x).$$

Stationary payoffs do not imply constant boundaries when time remaining changes.

At a deadline there is no option to acquire more information.

### Example (A Degenerate State Process)

Let  $X_t = x + t$ ,  $r = 1$ , and  $v(x) = 1$  for  $x < 0$ ,  $v(x) = 2$  for  $x \geq 0$ . Then

$$V(x) = \begin{cases} \max\{1, 2e^x\}, & x < 0, \\ 2, & x \geq 0. \end{cases}$$

For  $-\log 2 < x < 0$ , wait until  $X$  reaches  $0$ ; otherwise stop immediately.

At  $b = -\log 2$ ,  $V$  is continuous and  $v$  is locally smooth.

But  $V'_-(b) = 0$  and  $V'_+(b) = 1$ .

No Brownian variation across the boundary to enforce tangency.

Later: viscosity solutions handle stopping equations without classical smoothness.

## Expected Exit Time: Justifying the Limit

### Proof

Put  $\tau = \tau_{a,b}$ . Optional sampling at  $\tau \wedge T$  gives

$$\sigma^2 \mathbb{E}_x[\tau \wedge T] = \mathbb{E}_x[(X_{\tau \wedge T} - x)^2] \leq \max\{(a - x)^2, (b - x)^2\}.$$

Monotone convergence yields  $\mathbb{E}_x \tau < \infty$ , hence  $\tau < \infty$  a.s. Bounded convergence on the right then gives

$$\begin{aligned} \sigma^2 \mathbb{E}_x \tau &= \frac{x - a}{b - a} (b - x)^2 + \frac{b - x}{b - a} (a - x)^2 \\ &= (x - a)(b - x). \end{aligned}$$

Finite expected duration is proved, not assumed in the calculation.

Uniform boundedness of the stopped state supplies the required limits.

## Optional Sampling Needs Conditions

For standard Brownian motion, let  $\tau = \inf\{t : B_t = 1\}$ .

$\tau < \infty$  a.s. and  $B_\tau = 1$ , yet  $B_0 = 0$ .

Why does optional sampling not give  $1 = 0$ ?

## Optional Sampling Needs Conditions

For standard Brownian motion, let  $\tau = \inf\{t : B_t = 1\}$ .

$\tau < \infty$  a.s. and  $B_\tau = 1$ , yet  $B_0 = 0$ .

Why does optional sampling not give  $1 = 0$ ?

Although  $\mathbb{E}[B_{t \wedge \tau}] = 0$  for every  $t$ , the stopped process is not uniformly integrable:

$$\lim_{K \rightarrow \infty} \sup_{t \geq 0} \mathbb{E}[|B_{t \wedge \tau}| \mathbf{1}_{\{|B_{t \wedge \tau}| > K\}}] \neq 0.$$

Also  $\mathbb{E}\tau = \infty$ : success is certain, but expected waiting cost is infinite.

Bounded stopping times, bounded stopped states, or square-integrable stopped integrals give useful sufficient conditions in our applications.

## Two Further Rules for Stochastic Integrals

**Associativity.** If  $M_t = \int_0^t H_s dB_s$ , then

$$\int_0^t K_s dM_s = \int_0^t K_s H_s dB_s,$$

for predictable integrands satisfying the relevant integrability conditions.

## Two Further Rules for Stochastic Integrals

**Associativity.** If  $M_t = \int_0^t H_s dB_s$ , then

$$\int_0^t K_s dM_s = \int_0^t K_s H_s dB_s,$$

for predictable integrands satisfying the relevant integrability conditions.

**Integration by parts.** For continuous Itô processes  $X, Y$ ,

$$X_t Y_t - X_0 Y_0 = \int_0^t X_s dY_s + \int_0^t Y_s dX_s + [X, Y]_t.$$

The additional term is the product of simultaneous innovations.

If  $X$  has diffusion coefficient  $b$  and  $Y$  has coefficient  $d$  against the same Brownian motion,  $[X, Y]_t = \int_0^t b_s d_s ds$ .

Discounting has finite variation, so its covariation with a diffusion is zero.